Ricci-flat metrics on non-compact Calabi-Yau manifolds

Dmitri Bykov

Ludwig-Maximilians-Universität (München)

& Steklov Mathematical Institute (Moscow)

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Part I. General facts.

This talk will be about Calabi-Yau threefolds $\mathcal{M}$

- Complex manifolds of complex dimension three: $\dim_{\mathbb{C}} \mathcal{M} = 3$
- Zero first Chern class: $c_1(\mathcal{M}) = c_1(K) = 0$
  (K is the canonical bundle = bundle of 3-forms $\Omega \propto f(z)\, dz_1 \wedge dz_2 \wedge dz_3$), i.e. there exists a non-vanishing holomorphic 3-form Ω
- Such manifolds are used for supersymmetric compactifications in supergravity ($\mathbb{R}^{3,1} \times \mathcal{M}$), and serve as backgrounds for brane constructions ($AdS_5 \times Y^5$)

Non-compact Calabi-Yau manifolds

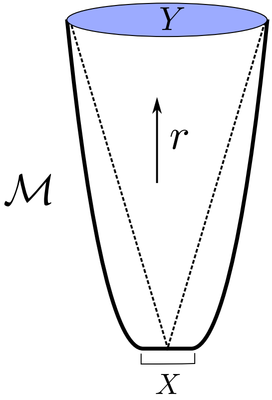
It is easy to show that compact Calabi-Yau's do not admit Killing vectors (apart from trivial cases), therefore explicit metrics are difficult to construct.

This talk will be about non-compact Calabi-Yau's, which **do** have symmetries. In this case the geometry of such manifolds may often be studied explicitly. These non-compact Calabi-Yau's may be thought of as describing singularities of compact Calabi-Yau's.

Let X be a positively curved complex surface, $c_1(X) > 0$. Here one should recall that $c_1(X) = \left[\frac{i}{2\pi} R_{m\bar{n}}\, dz^m \wedge d\bar{z}^{\bar{n}}\right] \in H^2(X, \mathbb{R})$. We will be studying the case

$$\mathcal{M} = \text{Total space of the canonical bundle of } X = \text{“Cone over } X\text{”}.$$

Non-compact Calabi-Yau manifolds



The corresponding singularity is pointlike and may be then resolved by gluing in a copy of X.

This is just like the prototypical $\mathbb{C}^2/\mathbb{Z}_2$-singularity ("$A_1$-singularity") given by equation $xy = z^2$ may be resolved by gluing in a copy of $\mathbb{CP}^1$ at the origin. The metric on the resolved space is then the Eguchi-Hanson metric. (However, this corresponds to $\mathcal{M}$ of complex dimension 2.)

First example. Calabi's ansatz.

If X admits a Kähler-Einstein metric, the metric on $\mathcal{M}$ may be found by means of an ansatz Calabi ('79)

$$\mathcal{K} = \mathcal{K}(|u|^2\, e^K),$$

where $\mathcal{K}$ and K are the Kähler potentials of $\mathcal{M}$ and X respectively. The Ricci-flatness equation becomes in this case an ODE for the function $\mathcal{K}(x)$.

For example, for $X = \mathbb{CP}^2$ one obtains in this way the (generalized) Eguchi-Hanson metric. Eguchi, Hanson ('78)

These metrics are asymptotically-conical, i.e. they have the form

$$ds^2 = dr^2 + r^2\,(\widetilde{ds^2})_Y \qquad \text{at} \qquad r \to \infty,$$

where $(\widetilde{ds^2})_Y$ is a Sasaki-Einstein metric on a 5D real manifold Y.

Calabi's ansatz.

An important characteristic of a Kähler metric on $\mathcal{M}$ is the cohomology class $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$ of the Kähler form. Since $\mathcal{M}$ is a total space of a line bundle, its cohomology is the same as that of the underlying surface X. Therefore, for instance for $X = \mathbb{CP}^2$ we have $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}$, but for $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ we have $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$.

Calabi's ansatz gives a metric with a very particular and fixed $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$. It turns out that $[\omega] \in H^2_c(\mathcal{M}, \mathbb{R}) \subset H^2(\mathcal{M}, \mathbb{R})$, where H^2_c is the compactly supported cohomology. By Poincaré duality, the group $H^2_c(\mathcal{M}, \mathbb{R}) \simeq H_4(\mathcal{M}, \mathbb{R}) = H_4(X, \mathbb{R}) = \mathbb{R}$ is one-dimensional.

The Calabi-Yau theorem.

The Calabi-Yau theorem Calabi ('57), Yau ('79) states, however, that, at least for compact $\mathcal{M}$, there is a unique Ricci-flat metric in **every** Kähler class $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$.
For the case of interest $\mathcal{M}$ is not compact, but asymptotically-conical, and in this case there exists a proposal for a CY theorem due to van Coevering ('2008), Goto (2012). Moreover, one has the decay estimates

$$|g - g_0|_{g_0} = O\left(\frac{1}{r^6}\right) \qquad \text{for} \quad [\omega] \in H_c^2(\mathcal{M}, \mathbb{R})$$

$$|g - g_0|_{g_0} = O\left(\frac{1}{r^2}\right) \qquad \text{for} \quad [\omega] \in H^2(\mathcal{M}, \mathbb{R}) \setminus H_c^2(\mathcal{M}, \mathbb{R}),$$

where g_0 is the conical metric. Such estimates were introduced for the case of ALE-manifolds in Joyce ('99).

Example. $X = \mathbb{CP}^1 \times \mathbb{CP}^1$.

The theory just described can be tested explicitly at the example of $X = \mathbb{CP}^1 \times \mathbb{CP}^1$. The ansatz for the Kähler potential on the cone over X is a generalized ansatz of Calabi constructed by Candelas, de la Ossa ('90), Pando Zayas, Tseytlin ('2001):

$$\mathcal{K} = a\,\log(1+|w^2|) + \mathcal{K}_0\left(|u^2|(1+|w^2|)(1+|x^2|)\right)\,.$$

The resulting metric, indeed, has two parameters that define the cohomology class of the Kähler form $[\omega] \in H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$. These correspond to the sizes of the two spheres. The relevant Sasakian manifold Y at $r \to \infty$ is the conifold $T^{11} = \frac{SU(2)\times SU(2)}{U(1)}$, and the decay at infinity agrees with the predicted one.

Part II. The del Pezzo surface of rank one.

The del Pezzo surface

We will be interested in the next-to-simplest example:

$X =$ del Pezzo surface of rank one

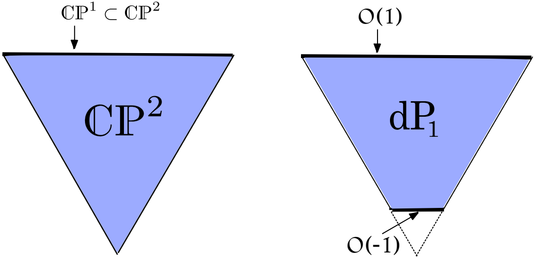
$(=$ Hirzebruch surface of rank one$) =$ the blow-up of $\mathbb{CP}^2$ at one point.

Pasquale del Pezzo (1859-1936),
Rector of the University of Naples,
Mayor of Naples, Senator

Del Pezzo surfaces ('1887) are natural generalizations to higher complex dimensions of positively curved Riemann surfaces (the sphere $S^2 = \mathbb{CP}^1$) and thus are very special.

Metrics on the del Pezzo surface

A blow-up means that we replace one point in $\mathbb{CP}^2$ by a sphere $\mathbb{CP}^1$. This $\mathbb{CP}^1$ 'remembers the direction', at which we approach the point. A 'good' metric on the new manifold should have two parameters, which describe the original size of the $\mathbb{CP}^2$ and the size of the glued in sphere $\mathbb{CP}^1$. The del Pezzo surface is a toric manifold, and the best way to think of it is via its moment polygon.



Metrics on the cone and toric geometry

A theorem of Tian, Yau ('87) says that there does **not** exist a Kähler-Einstein metric on $\mathbf{dP}_1$. How do we then construct a metric on the cone $\mathcal{M}$ over $\mathbf{dP}_1$? The only hope is to use its symmetries, which are those symmetries of $\mathbb{CP}^2$ that remain after the blow-up.

[Note: in fact, even a would-be Kähler-Einstein metric on the surface would not be sufficient to build the most general Ricci-flat metric on the cone]

The relevant isometry group is $U(1) \times U(2)$, however for the moment let us focus on the toric $U(1)^3$ subgroup. Generally, the Kähler potential has the form

$$\mathcal{K} = \mathcal{K}\left(\underbrace{|z_1|^2}_{=e^{t_1}}, \underbrace{|z_2|^2}_{=e^{t_2}}, \underbrace{|z_3|^2}_{=e^{t_3}}\right) .$$

Metrics on the cone and toric geometry

It is customary to introduce the symplectic potential $\mathcal{G}$ – the Legendre transform of the Kähler potential w.r.t. t_i:

$$\mathcal{G}(\mu_1, \mu_2, \mu_3) = \sum_{j=1}^{3} \mu_i \, t_i - \mathcal{K}$$

Here $\mu_i = \frac{\partial \mathcal{K}}{\partial t_i}$ are the moment maps for the $U(1)^3$ symmetries of the problem. The metric on $\mathcal{M}$ has the form

$$ds^2 = \frac{1}{4} \, \mathcal{G}_{ij} d\mu^i d\mu^j + (\mathcal{G}^{-1})^{ij} d\phi_i d\phi_j \, .$$

The Riemann tensor with all lower indices looks as follows:

$$R_{\bar{m}jk\bar{n}} = -\sum_{s,t} \, \mathcal{G}^{-1}_{ns} \, \frac{\partial^2 \mathcal{G}^{-1}_{jk}}{\partial \mu_s \partial \mu_t} \, \mathcal{G}^{-1}_{tm} \, .$$

Metrics on the cone and toric geometry

The domain, on which $\mathcal{G}$ is defined, is the moment polytope. The potential $\mathcal{G}$ has singularities at the boundaries of the polytope. For instance, for flat space $\mathbb{C}^3$ the polytope is the octant, and $\mathcal{G}$ has the form

$$\mathcal{G}_{\text{flat}} = \sum_{k=1}^{3} \mu_k \left(\log \mu_k - 1\right).$$

In general, at a boundary $L = 0$ the potential behaves as
$\mathcal{G} = L\left(\log L - 1\right) + \ldots$

Quite generally, Kähler metrics on toric manifolds were constructed by Guillemin ('94). They are built using Kähler quotients, and the corresponding symplectic potential exhibits the singularities just described.

In our problem we have more symmetry: $U(1) \times U(2)$ instead of $U(1)^3$. The Kähler potential is

$$\mathcal{K} = \mathcal{K}\left(\underbrace{|w|^2}_{=e^t}, \underbrace{|z_1|^2 + |z_2|^2}_{=e^s}\right),$$

which means that the metric is of cohomogeneity-2. For $\mathcal{G}$ this implies the following form:

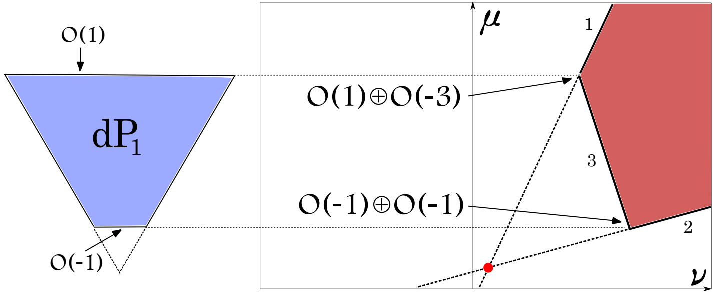
$$\mathcal{G} = \left(\frac{\mu}{2} + \tau\right) \log\left(\frac{\mu}{2} + \tau\right) + \left(\frac{\mu}{2} - \tau\right) \log\left(\frac{\mu}{2} - \tau\right) - \mu \log \mu + G(\mu, \nu)$$

$$\mu = \mu_1 + \mu_2, \qquad \tau = \frac{\mu_1 - \mu_2}{2}, \qquad \nu = \mu_3\,.$$

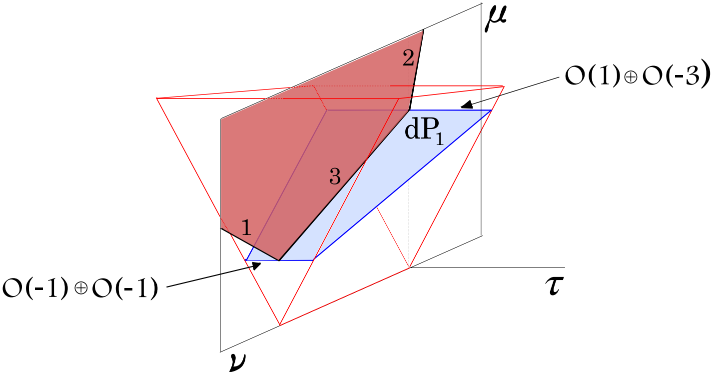
The Ricci-flatness equation is then a Monge-Ampère equation in two variables:

$$e^{G_\mu+G_\nu}\left(G_{\mu\mu}G_{\nu\nu}-G_{\mu\nu}^2\right)=\mu$$

The domain of definition is the moment polytope of the cone $\mathcal{M}$:



The full 3D moment polytope



The asymptotic behavior of G

One can construct an **exact** solution of the above equation taking the conical ansatz for the metric $ds^2 = dr^2 + r^2\, \widetilde{ds^2}$. We make a change of variables $(\mu, \nu) \to (\nu, \xi = \frac{\mu}{\nu})$ and look for G in the form ($\nu \propto r^2$)

$$G = 3\,\nu\,(\log\nu - 1) + \nu\,P(\xi)$$

One obtains an ODE for $P(\xi)$ that can be solved exactly. As a result,

$$G = \sum_{i=0}^{2} \frac{\mu - \xi_i\,\nu}{1 - \xi_i}\;(\log{(\mu - \xi_i\,\nu)} - 1)\,,$$

where ξ_i are the roots of $Q(\xi) = \xi^3 - \frac{3}{2}\,\xi^2 + d$. Varying d, one arrives at the Sasakian manifolds called $Y^{p,q}$ discovered in Gauntlett, Martelli, Sparks, Waldram ('2004). The topology of the underlying del Pezzo surface forces us to pick $Y^{2,1}$.

Uniqueness

The conical metric constructed above is singular at $r = 0$. Constructing a smooth – resolved – metric is rather difficult. For the moment let us assume that, for a fixed moment polytope, we constructed one such metric with potential G_0. To check uniqueness, one can expand $G = G_0 + H$ to first order in H:

$$\triangle_{G_0} H = 0 \quad \Rightarrow \quad 0 = \int d\mu\, d\nu\; H \triangle_{G_0} H \overset{?}{=} -\int d\mu\, d\nu\; (\nabla H)^2$$

Whether we may integrate by parts depends on the behavior at infinity, where we have asymptotically

$$\triangle_{G_0} H = 0 \quad \to \quad -\frac{\partial}{\partial \xi}\left(Q(\xi)\,\frac{\partial H}{\partial \xi}\right) + \frac{\xi}{\nu}\,\frac{\partial}{\partial \nu}\left(\nu^3\,\frac{\partial H}{\partial \nu}\right) = 0$$

Uniqueness

Substituting $H = \nu^m h(\xi)$, we get a Heun equation

$$-\frac{d}{d\xi}\left(Q(\xi)\,\frac{dh}{d\xi}\right) + m(m+2)\,\xi\,h(\xi) = 0$$

Therefore one needs to estimate the spectrum of the Laplacian on $Y^{2,1}$. We have the following result:

Proposition. [DB]

For the smallest non-zero eigenvalue λ of the Laplacian $\triangle_\xi = -\frac{d}{d\xi}\left(Q(\xi)\,\frac{dh}{d\xi}\right)$, entering the equation $\triangle_\xi f + \lambda\,\xi\,f = 0$, one has the lower bound $\lambda \geq 3$.

As a result, we obtain uniqueness of the metric for a given moment polytope. Therefore all potential moduli of the metric have to be related to the moduli of the polytope, which in turn are the Kähler moduli.

Part III. Killing-Yano forms.

Killing-Yano forms.

One approach to the explicit construction of a metric is to require that it admit a conformal Killing-Yano form (CKYF).

$$\begin{aligned}
\nabla_i \xi_j = 0 \quad &\Rightarrow \quad \text{Reduced holonomy} \\
\nabla_i \xi_j - \nabla_j \xi_i = 0 \quad &\Rightarrow \quad \xi = d\chi \\
\nabla_i \xi_j + \nabla_j \xi_i = 0 \quad &\Rightarrow \quad \text{Killing vector}
\end{aligned}$$

The Killing-Yano form $\omega_{ij}\, dx^i \wedge dx^j$:

$$\nabla_i \omega_{jk} + \nabla_j \omega_{ik} = 0$$

Conformal Killing-Yano form:

$$\nabla_i \omega_{jk} + \nabla_j \omega_{ik} - \text{trace parts} = 0$$

Killing-Yano forms. Some applications.

For the geodesic equation Killing tensors imply the existence of conserved quantities, polynomial in momenta (hence not directly related to isometries)
$\Rightarrow$ Integrability of the geodesic equation in the Kerr metric
Carter ('68), Walker, Penrose ('70), Floyd ('73), Teukolsky (2014)

[Note: The variables in the equation for linearized perturbations around Kerr separate for the same reason and lead to the Heun equation as well]

Many of the known (higher-dimensional) black hole solutions (Myers-Perry, etc.) admit Killing-Yano tensors
(Santillan (2012), Chervonyi, Lunin (2015), Frolov, Krtous, Kubiznak (2017))

Killing-Yano tensors in the context of Kähler geometry
Moroianu, Semmelmann (2002)

Killing-Yano forms.

On a Kähler manifold we may expand $\omega = \omega^{(2,0)} \oplus \omega^{(1,1)} \oplus \omega^{(0,2)}$. Especially simple is the situation when ω is Hermitian, i.e. $\omega^{(2,0)} = 0$. Introducing the 'shifted' form $\Omega_{a\bar{b}} = \omega_{a\bar{b}} - h\, g_{a\bar{b}}$ ($h = g^{a\bar{b}}\omega_{a\bar{b}}$), one gets the equation Apostolov, Calderbank, Gauduchon ('2002)

$$\nabla_a \Omega_{b\bar{c}} = -2 g_{a\bar{c}}\, \partial_b h$$

The tensor Ω has various names, such as Hamiltonian two-form, twistor form, etc. One can show that its eigenvalue functions x_i have orthogonal gradients. They can be related to 'moment map' variables μ_i corresponding to holomorphic isometries via the interesting formula:

$$\prod_{k=1}^{n} (\vartheta - x_k) = \sum_{k=0}^{n} \vartheta^k \, \mu_{k+1}.$$

The orthotoric metric.

At the end of the day the metric admitting a tensor Ω has the form (we set $x_1 = x, x_2 = y$, then $\mu = xy,\ \nu = x + y$)

$$ds^2 = \underbrace{x\,y}_{=\mu}\, g_{\mathbb{CP}^1} + (x-y)\,\left(\frac{dx^2}{P_1(x)} + \frac{dy^2}{P_2(y)}\right) + \text{angular part}$$

We call this metric the 'orthotoric metric'. We see that the variables separated. The requirement of Ricci-flatness fixes the functions P_1, P_2 to be cubic polynomials (one of which we encountered before):

$$P_1(x) = x^3 - \frac{3}{2}\,x^2 + c \qquad P_2(y) = y^3 - \frac{3}{2}\,y^2 + d\,.$$

The domain is $x \leq x_{min},\ y \in [y_1, y_2]$.

(Observe also the relation to the metrics of Pedersen, Poon ('1991).)

The orthotoric metric.

If we further require that the topology is that of the cone over $\mathbf{dP}_1$, the constants c and d are uniquely fixed. This metric was also obtained by Chen, Lü, Pope ('2006), Oota, Yasui ('2006) and was extensively studied by Martelli, Sparks ('2007).

The point is that the requirements of

(a) Ricci-flatness

(b) Cone over $\mathbf{dP}_1$ topology

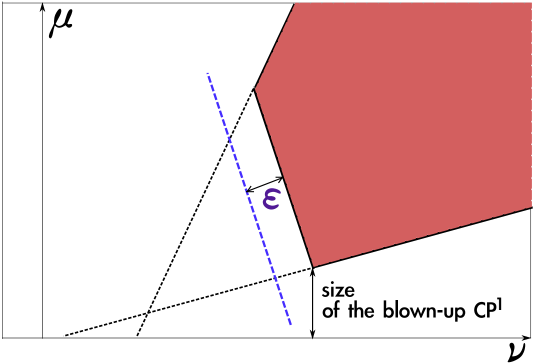
(c) CKYF of type $(1, 1)$

completely fix the metric.

According to the CY theorem, however, the metric should contain additional parameters, corresponding to the deformation of the moment polytope. Altogether there are 2 parameters, since $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$.

Deformation of the metric and the CKYF.

One parameter is somewhat 'trivial', as it corresponds to a rescaling of the metric. We can still look for the other non-trivial parameter, which corresponds to the following deformation:



Deformation of the metric and the CKYF.

In the equation $\triangle_{G_0} H = 0$, if we substitute the orthotoric potential G_0, variables separate:

$$\frac{1}{x}\frac{\partial}{\partial x}\left(P_1(x)\frac{\partial H}{\partial x}\right) - \frac{1}{y}\frac{\partial}{\partial y}\left(P_2(y)\frac{\partial H}{\partial y}\right) = 0$$

The unique solution compatible with the deformation of the moment polytope is

$$H(x,y) = \epsilon \int\limits_x^\infty \frac{d\hat{x}}{P_1(\hat{x})}\,.$$

For large x one has $H(x,y) = \frac{\epsilon}{2x^2} + \ldots$, and for the metric this implies $|g - g_0|_{g_0} = O\left(\frac{1}{r^6}\right)$. This implies that the variation of the Kähler form has the property $[\delta\omega] \in H_c^2(\mathcal{M}, \mathbb{R})$.

Deformation of the metric and the CKYF.

The next question is: what happens to the Killing-Yano form?

If it is deformed, it must acquire a non-zero $(2,0)$ part, i.e. $\omega^{2,0} = \omega_{mn}\, dz^m \wedge dz^n \neq 0$. On a Calabi-Yau manifold, one has a nowhere vanishing three-form $\Omega_{mnp}\, dz^m \wedge dz^n \wedge dz^p$, and one can use it to dualize $\omega^{2,0}$ and obtain a vector field $\omega^p := \widetilde{\Omega}^{mnp}\, \omega_{mn}$.

Using that $\mathcal{M}$ is Ricci-flat, we can show that ω^p has to satisfy a rather stringent requirement

$$R^n_{\ \ mp\bar{k}}\, \omega^p = 0\,. \tag{1}$$

As we mentioned earlier, on a toric manifold the curvature tensor is $R_{\bar{m}jk\bar{n}} = -\sum_{s,t} \mathcal{G}^{-1}_{ns}\, \frac{\partial^2 \mathcal{G}^{-1}_{jk}}{\partial \mu_s \partial \mu_t}\, \mathcal{G}^{-1}_{tm}$. Using the explicit expression for the orthotoric potential $\mathcal{G}$, we can show that the only solution is $\omega^p = 0$.

Proposition. [DB]

There exists a first-order deformation of the orthotoric metric that preserves Ricci-flatness and corresponds to a deformation of the moment polytope. Moreover, the deformation of the Kähler form has the property $[\delta\omega] \in H_c^2(\mathcal{M}, \mathbb{R})$. The deformed metric does not possess a conformal Killing-Yano tensor.

Summary.

- Metrics on non-compact Calabi-Yau manifold can be sometimes constructed explicitly
- Examples in $\dim_{\mathbb{C}}\mathcal{M} = 3$: Cones over $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$
- More complicated cases with conformal Killing-Yano tensors
- In the case of the cone over $\mathbf{dP}_1$ the corresponding metric is not the most general one, predicted by the CY theorem
- One can explicitly construct a first-order deformation
- What is the significance of the explicitly known (orthotoric) metric? Can one obtain a closed expression for the metric in the general case, or in other special cases?