α' adventures

with Jim Liu (in progress + [arXiv:1304.3137])

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with Ilarion Melnikov and Savdeep Sethi ([arXiv:1403.4298])

"If a person has lived through war, poverty and love, he has lived a full life" (O. Henry)

if not ... computing higher derivative terms is the closest you get

Higher-derivative terms

- Probe the stringy regime (α' and genus expansion)
- Needed for consistency
- Important for applications

▷ The plan:

- more α' (... from dualities)
- heterotic generalised geometry
- supersymmetric heterotic backgrounds

▷ Main equation (for this talk) - Heterotic Bianchi Identity :

$$dH_3 = \frac{\alpha'}{4}(\text{tr}\, R^2(\Omega_+) - \text{tr}\, F^2) + \mathcal{O}(\alpha')$$

where

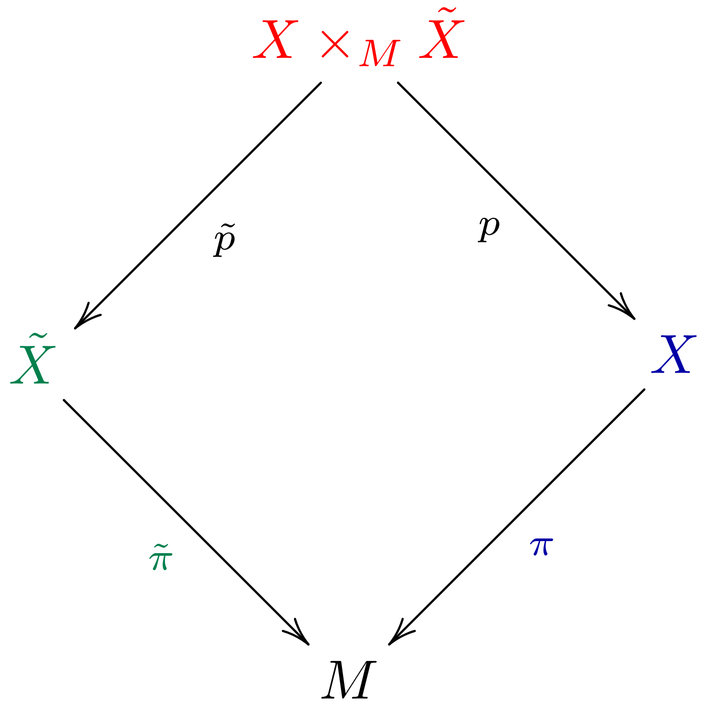
$$R(\Omega_\pm) = R(\Omega^{\rm LC} \pm \frac{1}{2}\mathcal{H}) = R(\Omega) \pm \tfrac{1}{2} d\mathcal{H} + \tfrac{1}{4}\mathcal{H} \wedge \mathcal{H}, \qquad \mathcal{H}^{ab} = H_\mu{}^{ab} dx^\mu$$

- Choice of connection in heterotic BI
  - ▷ tied to a choice of field redefinitions in the higher order curvature corrections to supergravity
  - ▷ manifest (0,1) world-sheet supersymmetry: covariant Hermitian space-time metric $G \quad \leftrightarrow \quad \Omega_+ = \Omega^{\mathrm{LC}} + \frac{1}{2}\mathcal{H}$
    - ▷ (0,2) world-sheet SUSY likes Chern connection, but ... that requires G to pick up non-trivial space-time Lorentz and gauge transformations (and α' shifts in susy transformations)
  - ▷ (Narain) T-duality

- Lessons of α' expansion:
  - ▷ it is not physically correct to treat the heterotic space-time equations of motion, truncated to include just the leading order α' corrections, as a closed system
  - ▷ simultaneously consider the α' expansion for both the solution and the equations of motion

I. Dualities and higher derivative couplings

T-duality - coord-independent $O(n,n)$ transformation (perturbative symmetry) in a background with n isometries v^i leading to **topology change.** For type II theories:



Correspondence space $Y = X \times_M \tilde{X}$:

- ◇ a circle bundle over X with first Chern class $\pi^*(c_1(\tilde{X}))$
- ◇ a circle bundle over $\tilde{X}$ with first Chern class $\tilde{\pi}^*(c_1(\tilde{X}))(\mathcal{L}_{\mathsf{v}} H = 0 \;\Rightarrow\; \mathrm{d}(\imath_{\mathsf{v}} H) = 0)$

T-duality:

$$\pi_* H = c_1(\tilde{X}) \quad \tilde{\pi}_* \tilde{H} = c_1(X) \qquad \in H^2(M, \mathbb{Z})$$

- Start with $\mathrm{d}H = 0$ and

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_{10} \\ & & \downarrow \pi \\ & & X_9 \end{array}$$

$\mathcal{L}_v g_{10} = 0 = \mathcal{L}_v H$:

$$\mathrm{d}e = \pi^* T \qquad (\mathcal{L}_v\, e = 0)$$

$$H = \pi^* h_3 + \pi^* \tilde{T} \wedge e$$

$$\mathrm{d}H = 0 \qquad \Leftrightarrow \qquad \begin{cases} \bullet \quad \mathrm{d}h_3 = \tilde{T} \wedge T = \frac{1}{4}\left[(T_+)^2 - (T_-)^2\right] \\ \bullet \quad \mathrm{d}\tilde{T} = 0 \end{cases}$$

- $T_\pm = T \pm \tilde{T}$
- Locally $H = \mathrm{d}B = \mathrm{d}(b_2 + b_1 \wedge e) \qquad \Rightarrow \qquad \tilde{T} = \mathrm{d}b_1$ and $h_3 = \mathrm{d}b_2 - b_1 \wedge T$
- h_3 - invariant under T-duality (b_2 is not!)
- Note $b_2 \ \rightarrow \ b_2 + \mathrm{d}\lambda_1 + \lambda_0 T$, $b_1 \ \rightarrow \ b_1 + \mathrm{d}\lambda_0$ and h_3 is gauge invariant

- Now turn to $\mathrm{d}H = \frac{\alpha'}{4}[\text{tr}\, R^2(\Omega_+) - \text{tr}\, F^2]$

denote $X_4(\Omega_+, A) \equiv \text{tr}\, R^2(\Omega_+) - \text{tr}\, F^2 = \pi^* \tilde{X}_4 + \pi^* \tilde{X}_3 \wedge e$

$$\mathrm{d}X_4 = 0 \qquad \Leftrightarrow \qquad \begin{cases} \bullet \quad \mathrm{d}\tilde{X}_3 = 0 \\ \bullet \quad \mathrm{d}\tilde{X}_4 - \tilde{X}_3 \wedge T = 0 \end{cases}$$

If $\mathcal{L}_v X_3^{(0)} = 0 \qquad \Rightarrow \qquad \tilde{X}_3$ is exact: $\tilde{X}_3 = \mathrm{d}\tilde{X}_2$

$$\mathrm{d}H = \frac{\alpha'}{4} X_4(\Omega_+, A) \qquad \Leftrightarrow \qquad \begin{cases} \bullet \quad \mathrm{d}\tilde{H}_2 = \frac{\alpha'}{4}\tilde{X}_3 \quad \Rightarrow \quad \tilde{T} = \tilde{H}_2 - \frac{\alpha'}{4}\tilde{X}_2 \\ \bullet \quad \mathrm{d}h_3 = \frac{\alpha'}{4}\left[\tilde{X}_4 - \tilde{X}_2 \wedge T\right] - T \wedge \tilde{T} \end{cases}$$

$$\begin{aligned} -8\pi^2\left(\tilde{X}_4 - \tilde{X}_2 \wedge T\right) \quad = \quad & R^{\alpha\beta}(\omega_+) \wedge R^{\alpha\beta}(\omega_+) - \tfrac{1}{2} R^{\alpha\beta}(\omega_+) \wedge \hat{T}_+{}^{\alpha}{}_{\gamma} \hat{T}_+{}^{\beta}{}_{\delta} e^{\gamma} \wedge e^{\delta} \\ & + \tfrac{1}{2}(\nabla_\gamma \hat{T}_+{}^{\alpha}{}_{\delta} + \tfrac{1}{2} h_\gamma{}^{\alpha\rho} \hat{T}_+{}^{\rho}{}_{\delta})(\nabla_\epsilon \hat{T}_+{}^{\alpha}{}_{\iota} + \tfrac{1}{2} h_\epsilon{}^{\alpha\sigma} \hat{T}_+{}^{\sigma}{}_{\iota}) e^{\gamma} \wedge e^{\delta} \wedge e^{\epsilon} \wedge e^{\iota} \\ & + \mathcal{O}(\alpha') - F \wedge F \end{aligned}$$

problems?

- missing RF^2 and $(\nabla F)^2$ terms
- $\hat{T}_+$ (“graviphoton”) terms - OK with SUSY, but ...O(n) vs. O(n,n+16)
- $(\alpha')^2$ terms? $\quad \hat{T}_+ = T + \tilde{H}_2 = T_+ + \frac{\alpha'}{4}\tilde{X}_2$

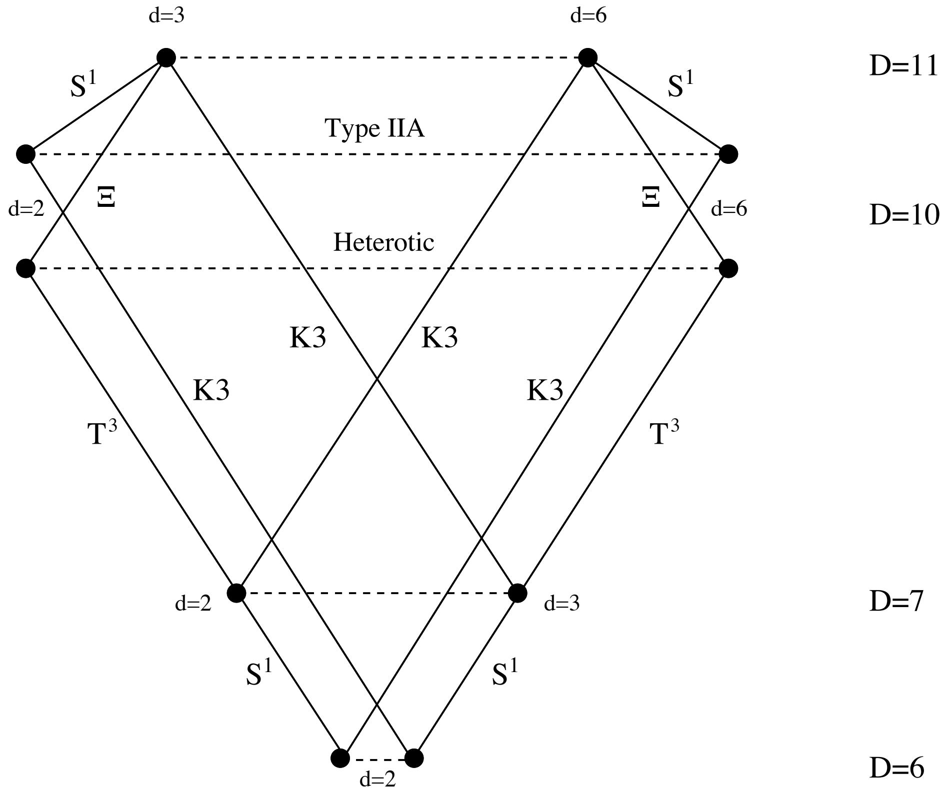
Computing on $\mathbb{T}^n$ vs $\mathbb{R}^n$ (generic point in moduli space)

- Apperance of couplings that vanish in decompactification limit
- $T_+^i \quad \leftarrow \quad V_I^i \mathcal{F}^I$ where $i = 1, \cdots, n$ and $I = 1, \cdots, 2n+16$
- V_I^i is $O(n, n+16)/O(n) \times O(n+16)$ (part of) coset element
  - ◇ $V = \{V_I^i; V_I^\alpha\}$ with $\alpha = 1, \cdots n+16$
  - ◇ $V_I^i = \mathbb{I}_n + \tilde{V}_I^i(\phi)$
  - ◇ T_+^i couplings start from 3-pt, the generic $\mathcal{F}^I$ - form 4-pt
- no $(\alpha')^2$ terms
- Het/M-th duality: three-level $B\mathcal{F}^2 \;\mapsto\; (CGG)_M$; $BR\mathcal{F}^2$ and $B(\mathcal{D}\mathcal{F})^2 \;\mapsto\; ???$

Reduction of heterotic GS couplings

- (on $K3$ these give rise to GS couplings in 6d $(1,0)$ theory $\sim \alpha'$)
- on $\mathbb{T}^n$ do not give rise to terms $\sim \alpha'$ but ...
- do **not** vanish! $\quad \Rightarrow \quad \imath_{\mathsf{v}_1} \ldots \imath_{\mathsf{v}_n}(B \wedge X^{\mathrm{GS}}) \neq 0$

Six-dimensional Heterotic/IIA duality



- $H^{\rm het} = e^{2\phi} * H^{\rm IIA}, \qquad g_{\mu\nu}^{\rm het} = e^{2\phi} g_{\mu\nu}^{\rm IIA}, \qquad \phi = -\varphi^{\rm IIA}$
- Het. BI : $dH_3 = \frac{\alpha'}{4}(\text{tr}\, R^2 - \text{tr}\, F^2) \quad \Leftrightarrow \quad$ Type II EOM ($\Leftarrow \; B \wedge (F^I \wedge F^J d_{IJ} - \text{tr}\, R^2)$)
  - ◇ $B \wedge F \wedge F$ descends from 11d $C_3 \wedge G_4 \wedge G_4 \quad$ ($d_{iJ} = \int_{K3} \omega_I \wedge \omega_J$)
  - ◇ $B \wedge \text{tr}\, R^2$ descends from 11d $C_3 \wedge X_8(\Omega^{\rm LC})$
- Ω_+ and the duality?

Heterotic effective action

$$e^{-1}\mathcal{L} = e^{-2\phi}[R + 4\partial\phi^2 - \tfrac{1}{12}H^2_{\mu\nu\rho} - \tfrac{1}{4}\alpha'\mathrm{tr}\, F^2_{\mu\nu} + \tfrac{1}{8}\alpha' R_{\mu\nu\lambda\sigma}(\Omega_+)R^{\mu\nu\lambda\sigma}(\Omega_+)],$$

Dirac operator in the susy transformations has $\Omega_- = \Omega - \frac{1}{2}\mathcal{H}$ (similar to the sign flip in the local expressions for index theorems for Dirac operation with $\Omega \neq \Omega^{\mathrm{L}C}$)

$$R(\Omega_+) = R(\Omega + \frac{1}{2}\mathcal{H}) = R(\Omega) + \tfrac{1}{2}d\mathcal{H} + \tfrac{1}{4}\mathcal{H}\wedge\mathcal{H}, \qquad \mathcal{H}^{ab} = H_\mu{}^{ab}dx^\mu.$$

H has a non-trivial Bianchi identity

$$dH = \tfrac{1}{4}\alpha'\mathrm{tr}\, R(\Omega_+)\wedge R(\Omega_+) - \tfrac{1}{4}\alpha'\mathrm{tr}\, F\wedge F.$$

The equations of motion (up to $(\alpha')^2$ terms) :

$$\begin{aligned}
R - 4\partial\phi^2 + 4\Box\phi - \tfrac{1}{12}H^2_{\mu\nu\rho} - \tfrac{1}{4}\alpha'\mathrm{tr}\, F^2_{\mu\nu} + \tfrac{1}{8}\alpha' R_{\mu\nu\lambda\sigma}(\Omega_+)R^{\mu\nu\lambda\sigma}(\Omega_+) &= 0,\\
R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - \tfrac{1}{4}\alpha'\mathrm{tr}\, F_{\mu\rho}F_\nu{}^\rho + \tfrac{1}{4}\alpha' R_{\mu\lambda\rho\sigma}(\Omega_+)R_\nu{}^{\lambda\rho\sigma}(\Omega_+) &= 0,\\
d(e^{-2\phi} * H) &= 0,\\
e^{2\phi}d(e^{-2\phi} * F) + A\wedge *F - *F\wedge A + F\wedge *H &= 0
\end{aligned}$$

Two $(1,1)$ supergravities in 6D:

- $(1,1)_{\text{het}}$: Non-trivial BI and single Dirac operator
- $(1,1)_{\text{IIA}}$: $\mathrm{d}H = 0$ and two Dirac operators $\partial + \Omega_\pm \to D_\pm$

Dualisation of $\mathrm{d}H = \frac{1}{4}\alpha' \mathrm{tr}\, R(\Omega_+) \wedge R(\Omega_+) + \ldots = 2\pi^2\alpha'[\overline{X_4} + \underline{X_4}]$:

- $\overline{X_4}$(het) $\longrightarrow$ $\overline{X_4}$(IIA) (up to terms vanishing on-shell)
- $\underline{X_4}$(het) $\longrightarrow$ CP-even terms in IIA

On IIA side: $\mathrm{d}(e^{2\phi} * H + \alpha' * T) = 2\pi^2\alpha'\overline{X_4}$

- $\alpha'\mathrm{d} * T$ comes from the variation of (one-loop) $\epsilon_4\epsilon_4 R^2$ and $t_4 t_4 R^2$ terms in 6D IIA theory (lin. duality: $H = 0 \quad \Rightarrow \quad T \to 0 \quad \& \quad \overline{X_4} \to X_4$)
- CS term in $(1,1)_{\text{IIA}}$: $\sim \alpha' B_2 \wedge \overline{X_4}$

$$2e^{-1}\delta\mathcal{L}_{\text{CP-even}} = \begin{cases} R_{\mu\nu\rho\sigma}(\Omega_+)^2 + E_4(\Omega_+) + 4R_{\mu\nu\rho\sigma}(\Omega_+)H^{\mu\rho a}H^{\nu\sigma a} - 4R_{\mu\nu}(\Omega_+)H^{2\,\mu\nu} \\ +\frac{2}{3}R(\Omega_+)H^2 + \frac{1}{9}(H^2)^2 - \frac{2}{3}H^4 \end{cases}$$

- Matched by 4-pt function calculation! **NO** $\mathcal{O}(\alpha')$ corrections to 6d duality dictionary

Summary of type II $(\alpha')^3$ couplings (10D):

	No B	With B
e-o **+** **o-e**	$\frac{1}{8}(t_8\epsilon_{10}+\epsilon_{10}t_8)BR^4$ $= B\wedge X_8(\Omega^{\rm LC})$ $= \frac{1}{192(2\pi)^4}B\wedge\left(\text{tr}\,R^4-\frac{1}{4}(\text{tr}\,R^2)^2\right)$	$\frac{1}{8}(t_8\epsilon_{10}+\epsilon_{10}t_8)BR^4(\Omega_+)$ $=\frac{1}{8}t_8\epsilon_{10}B(R^4(\Omega_+)+R^4(\Omega_-))$ $=\frac{1}{2}B\wedge[X_8(\Omega_+)+X_8(\Omega_-)]$ $=\frac{1}{192\cdot(2\pi)^4}B\wedge\left(\text{tr}\,R^4-\frac{1}{4}(\text{tr}\,R^2)^2+\text{exact terms}\right)$
e-e	$t_8t_8R^4$	$t_8t_8R^4(\Omega_+)=t_8t_8R^4(\Omega_-)$
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	$\frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_+)^4+\frac{8}{3}H^2R(\Omega_+)^3+\cdots\right)$ $=\frac{1}{8}\epsilon_{10}\epsilon_{10}\left(R(\Omega_-)^4+\frac{8}{3}H^2R(\Omega_-)^3+\cdots\right)$

- ◇ Connection with torsion: ${\Omega_\pm}_\mu{}^{\alpha\beta}=\Omega_\mu{}^{\alpha\beta}\pm\frac{1}{2}H_\mu{}^{\alpha\beta}$ (where $\mathcal{H}^{\alpha\beta}=H_\mu{}^{\alpha\beta}dx^\mu$)
- ◇ Curvature: $R(\Omega_\pm)=R\pm\frac{1}{2}d\mathcal{H}+\frac{1}{4}\mathcal{H}\wedge\mathcal{H}$
- ◇ New kinematic structures in o-o sector

Couplings can be lifted to eleven dimensions

- All expressions even in H: $H^2 \to G^2$ with an extra pair of summed indices
- lifting ambiguities: more terms in eleven dimensions than in ten

Reduction of heterotic GS couplings on $\mathbb{T}^3$ vs. reduction of $C_3 \wedge X_8$ on $K3$

$$B \wedge X^{\rm GS}(R,F) \quad \Rightarrow \quad \left\{ \begin{array}{l} \bullet \; d_{IJKLM} a^I \wedge {\rm d}a^J \wedge F^K \wedge F^L \wedge F^L \Rightarrow 0 \\ \diamond \; d_{IJK} F^I \wedge R_2^{ab} \wedge (F_1)_a^J \wedge \nabla (F_1)_b^K \\ \diamond \; \tilde{d}_{IJK} F^I \wedge R_2^{ab} \wedge R_2^{cd} \wedge F_{ac}^J \wedge \nabla F_{bd}^K + \cdots \end{array} \right\} \quad \Leftarrow \quad C \wedge \overline{X_8(\Omega, G)}$$

- 0 for four-derivative terms at generic lattice $\Gamma_{3,19}$ points. Non-zero at enhancement points $\quad \Leftarrow \quad$ *singular* $K3$ surfaces
- $C_3 \wedge X_8$ cannot give rise to four derivative terms beyond $C \wedge$ tr R^2
  - ▷ $R\mathcal{F}^2$ and $(\mathcal{DF})^2$ terms in heterotic BI are matched by $\overline{X_8(\Omega, G)}$

◇ d_{IJK}, $\tilde{d}_{IJK}$ are computable on heterotic side

◇ on M-theory involve integrals dependent on $K3$ *metric* (e.g. $\int \omega_2^I \wedge \omega_{ab}^J R_2^{bc} \omega_{ca}^K$)

◇ $C_3 \wedge X_8(\Omega, G)$ gets fixed!

II. Generalised geometry for heterotic strings

Generalised complex structure (GCG)

- GCG $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$ $\quad (\mathcal{J}^2 = -1; \quad \mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I})$
  - ◇ Structure group: $\Rightarrow$ U(3,3)
- GCS integrable: $\pi_+[\pi_-(v), \pi_-(w)]_{\text{Lie}} = 0 \;\mapsto\; \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$ with Courant bracket:

$$[v + \xi,\, w + \eta] = [v, w]_{\text{Lie}} + \left\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \tfrac{1}{2}\, \mathrm{d}(\imath_v \eta - \imath_w \xi) \right\}$$

(Courant closes on $L_{\mathcal{J}}$ – the i-eigenbundles of $\mathcal{J}$.)

- Closed B-transform $(v_1, \rho_1) \mapsto e^B(v_1, \rho_1) = (v_1, \rho_1 + \imath_{v_1} B)$ is an auto-morphism of Courant : $[e^B(v_1, \rho_1), e^B(v_2, \rho_2)] \mapsto e^B[(v_1, \rho_1), (v_2, \rho_2)]$
- Twisting: $\quad \mathrm{d} \mapsto \mathrm{d} - H\wedge, \qquad [.,.]_C \mapsto [.,.]_C + \underline{\imath_v \imath_w H}$
- Replacing Lie bracket by Courant allows to extend Riemannian objects to generalised objects (e.g. generalised connection)

Generalised tangent bundle :

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0\,,$$

Sections of E:

$$X = \begin{pmatrix} v \\ \xi \end{pmatrix} \qquad \longmapsto \qquad X' = e^{-B} X = \begin{pmatrix} \mathbb{I} & 0 \\ -B & \mathbb{I} \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ \xi - \imath_v B \end{pmatrix}.$$

- Note $(v, \xi) \to (v, \xi - \imath_v \mathrm{d}\Lambda)$
- Courant on E $\quad\to\quad$ *twisted* Courant on $T \oplus T^*$

On $L_\phi \otimes E$ (ϕ , g and B define $S^{\pm}(E) \cong L_\phi \otimes \Lambda^{\text{even/odd}} T^* M$):

- Courant does not define an unambiguous gen. Riemman but ...unique
  $\hat{R}_{ab} = R_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab})$ and
  $\hat{R} = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 \qquad \leftrightarrow \qquad$ LBT: $S^{\mathrm{NS}}\epsilon = 4\left(D^a D_a - D^2\right)\epsilon$
- The dynamics and supersymmetry transformations of type II supergravity theories are captured by a (torsion-free) generalised connection

- α' corrections to geometry
- HS ($\mathrm{d}H \neq 0$): $\Rightarrow$ $B \to B + \mathrm{d}\Lambda + \frac{\alpha'}{4}\mathrm{d}^{-1}\delta\mathrm{d}^{-1}X_4(\Omega_+, A)$

$\Bigg\}$ $\leftarrow$ Extensions of GTB

Two simple ideas:

- Find an extended gen. tangent space with a closed 3-form! (*Global data*)

◇ Generalise $U(1)$ fibration $S^1 \hookrightarrow X \xrightarrow{\pi} M$ case:

▷ $\mathrm{d}H = 0$ on X $\Rightarrow$ $\mathrm{d}h_3 = \tilde{T} \wedge T = \frac{1}{4}\left[(T_+)^2 - (T_-)^2\right]$ on M

▷ h_3 is gauge invariant!

Generalised heterotic tangent space is built as a double fibration:

$$0 \longrightarrow \mathfrak{g} \longrightarrow C \longrightarrow TM \longrightarrow 0\,,$$

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow C \longrightarrow 0$$

◇ $\mathfrak{g}$ is the adjoint bundle given a principle G-bundle

◇ Locally $E \simeq TM \oplus T^*M \oplus \mathfrak{g}$

◇ Obstruction: $p_1(\mathfrak{g}) = 0$

gen. Lichnerowicz theorem (LBT) $\Rightarrow$ effective actions (*Local data*)

- (gen.) Lichnerowicz theorem: $\left(D^A D_A - D^2\right)\epsilon = \left[\frac{1}{4}S + \gamma^{abcd} I_{abcd}\right]\epsilon$ (S tensorial!)

◇ Heterotic effective action: $S = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12}H^2 - \frac{\alpha'}{4}\mathrm{tr}\,\hat{\mathcal{F}}^2$

◇ $I_{abcd} = \frac{1}{6}\nabla_{[a}H_{bcd]} - \frac{\alpha'}{8}\mathrm{tr}\,\hat{\mathcal{F}}_{[ab}\hat{\mathcal{F}}_{cd]} = 0$

◇ $\left.\begin{array}{l} \delta\psi_a = D_a\epsilon = \nabla_a\epsilon + \frac{1}{8}H_{abc}\gamma^{bc}\epsilon \\ \delta\zeta_\alpha = D_\alpha\epsilon = -\frac{1}{8}\sqrt{2\alpha'}\hat{\mathcal{F}}_{ab\alpha}\gamma^{ab}\epsilon \end{array}\right\}$ $\leftarrow$ covariant derivative ($A = \{a, \alpha\}$)

◇ $\delta\lambda = D\epsilon = \left(\gamma^a\nabla_a + \frac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon$ $\leftarrow$ Dirac operator

Gravitational terms (obstruction to E) ?

◇ take $G \to G_{\mathrm{gauge}} \times O(n)$.... This splits $E = \tilde{C}_+ \oplus \tilde{C}_{\mathfrak{g}} \oplus \tilde{C}_-$

◇ reduce the structure group of E to $O(n) \times G \times O(n) \subset O(d + \dim(\mathfrak{g})) \times O(n)$

◇ Identify $O(n) \in G$ with $O(n)$ in $\tilde{C}_+$

▷ Works only for $\hat{A} = \Omega_- = \omega^{\mathrm{LC}} - \frac{1}{2}\mathcal{H}$!!! (cf susy for Ω_-)

▷ For type II $G \to O(n) \times O(n)$ does **NOT** work

▷ Flip of the sign in $\mathcal{O}(\alpha')$ effective action wrt D_a : $\Omega_+ \longrightarrow \Omega_-$!!!

◇ $R_{mnpq}(\Omega^-) - R_{pqmn}(\Omega^+) = -12 dH_{mnpq}$

- All orders in α':
  - ▷ "gaugino" $\psi_{ab} \in \Gamma(\Lambda^2 C_+ \otimes S(C_-))$ for "gauge group" $O(n)_+$
    - ◇ $\delta\psi_{ab} = \frac{1}{8}\sqrt{\alpha'} R(\Omega^-)_{\bar{a}\bar{b}ab}\gamma^{\bar{a}\bar{b}}\epsilon .... = D_{ab}\epsilon \ \ (?)$
  - ▷ ψ_{ab} - *composite* "gravitino curvature"
    - ◇ $\delta\psi_{ab} = D_{ab}\epsilon + \frac{1}{8}\sqrt{\alpha'}\,(3\alpha'[\text{tr}\, F \wedge F - \text{tr}\, R(\Omega^-) \wedge R(\Omega^-)]_{ab\bar{a}\bar{b}})\,\gamma^{\bar{a}\bar{b}}\epsilon \ \rightarrow \ \hat{D}_{ab}\epsilon$
  - ▷ $D_{ab} \ \rightarrow \ \hat{D}_{ab}$ in LBT $\Rightarrow$ $\mathcal{O}(\alpha'^2)$ modifications of susy
  - ▷ (iterative) hierarchy of higher α' corrections (consistent with GCG)

- Lichnerowicz-Bismut theorem $\Leftrightarrow$ Supersymmetry (Susy Ward identity)
  - ▷ susy modifications due to gaugings:
    - ◇ $\delta'_\epsilon\Psi = A \cdot \epsilon, \qquad \delta'_\epsilon\chi = B \cdot \epsilon$
  - ▷ susy Ward identity (for potential V):
    - ◇ $B^\dagger B - A^\dagger A = V\,\mathbb{I}$
    - ◇ 10d theory views as a reduction to zero dimensions on a 10d manifold M:
      - global sym. group $\mathcal{G}$ $\Leftrightarrow$ group of diffs and local $O(d,d) \times \mathbb{R}^+$ gauge transf.
      - R-symmetry $\mathcal{H}$ $\Leftrightarrow$ subgroup of local $O(d) \times O(d)$ gauge transf.
  - ▷ GCG $\Leftrightarrow$ infinite-dimensional version of the embedding tensor formalism

III. Torsional heterotic backgrounds

Conditions for supersymmetry (up to $\sim \mathcal{O}(\alpha'))$ on $M_4 \times X_6$:

- Internal space:

$$\omega^3 = \frac{3i}{4}\Omega \wedge \bar{\Omega}\,, \qquad \omega \wedge \Omega = 0$$

  space with trivial canonical bundle ($SU(3)$ structure)

- dilaton:

$$\mathrm{d}(e^{-\phi}\omega^2) = 0\,, \qquad d(e^{-\phi}\Omega) = 0$$

  internal manifold is complex

- H-flux:

$$H = i(\bar{\partial} - \partial)\omega = 0$$

- Gauge fields:

$$\mathcal{F} \wedge \Omega = \ 0 \ = \mathcal{F} \wedge \bar{\Omega}, \qquad \omega^2 \wedge \mathcal{F} = 0$$

  Hermitean YM

▷ Susy + Bianchi identity $\quad \Rightarrow \quad$ solution

Solutions

- ◇ Leading ($\sim \mathcal{O}(\alpha')$) corrections to the geometry in CY compactifications

- Theorem: for 4d $\mathcal{N}=2$ supersymmetry need internal CFT with with $c=9$ and $(0,4)$ susy + a pair of $U(1)$'s ($c=3$ with $(0,2)$ susy)
  - ▷ $\mathbb{T}^2$ fibration over hyper-Kähler base
  - ▷ Generic internal space for het. strings in $\mathcal{N}=2$ heterotic/type II duality (not $\mathbb{T}^2 \times K3$)
  - ▷ the target-space necessarily has string-scale cycles; flux solutions do not have a 10d large radius limit, they do have an *eight-dimensional* large radius limit
  - ▷ Duality Het/$M \times \mathbb{T}^2$ vs. F theory/$M \times K3_e$:
    - ◇ eight-dimensional theory (min. susy) with $O(2,18,\mathbb{Z})$ symmetry
    - ◇ Non-trivial $G_4 \quad \Rightarrow \quad$ nontrivial $\mathbb{T}^2 \hookrightarrow X \xrightarrow{\pi} M$
    - ◇ $G_4 = 0$ for $X = \mathbb{T}^4$, so X only be $K3$
    - ◇ F-theory: $G_4 = \gamma \wedge \gamma'$ (γ and γ' - $(1,1)$ primitive forms on M and $K3_e$)
  - ▷ $G_4 = \gamma \wedge \gamma' + \Omega_0 \times \bar{\Omega}'_0 + c.c.$ only $\mathcal{N}=1$ susy is preserved
    - ◇ eight-dimensional graviphotons are affected!
    - ◇ deformations of C.S. in $\mathcal{N}=2$ case either preserve all susy or break all susy

$\mathbb{T}^2 \hookrightarrow X \xrightarrow{\pi} K3$

- on the K3 base

$$\omega_0^2 = \frac{1}{2}\Omega_0\bar{\Omega}_0 \ , \qquad \omega_0\Omega_0 = \Omega_0^2 = 0$$

$(\omega_0, \Omega_0, \mathcal{A})$ - Calabi-Yau structure

- On X:

$$\omega_X = e^{2\phi}\omega_0 + \frac{i\mathbf{a}}{2}\Theta\bar{\Theta} \ , \qquad \Omega_X = e^{2\phi}\sqrt{\mathbf{a}}\Omega_0\Theta \ , \qquad \mathcal{F} = \pi^*\bar{\partial}\mathcal{A}$$

◇ $\boldsymbol{F} = \boldsymbol{F}^1 + i\boldsymbol{F}^2$ ($d\boldsymbol{\Theta}^{1,2} = \pi^*(\boldsymbol{F}^{1,2}),\ \boldsymbol{F}^{1,2} \in H^2(M, 2\pi\mathbb{Z})$) must satisfy:

$$\omega_0 \wedge \boldsymbol{F} = 0 \qquad \Omega_0 \wedge \boldsymbol{F} = 0$$

◇ Curvature of $\mathbb{T}^2$ bundle:

$$\boldsymbol{F} = F + F' \ , \qquad F \in H^{1,1}(M) \ , \qquad F' \in H^{2,0}(M)$$

▷ supersymmetry:

$$\left.\begin{array}{l} F' = 0 \quad (\text{i.e. } \bar{\Omega}_0 \wedge \boldsymbol{F} = 0) \\ F' \neq 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \mathcal{N} = 2 \ \text{(only left symmetries broken)} \\ \mathcal{N} = 1 \ \text{(broken right (susy!) symmetries)} \end{array}\right.$$

- H-flux:

$$H = i(\bar\partial - \partial)\omega_X = i\omega_0(\bar\partial - \partial)e^{2\phi} + \frac{\mathbf{a}}{2}(\bar F' - \bar F)\Theta + \frac{\mathbf{a}}{2}(F' - F)\boldsymbol{\Theta}$$
$$= H_{\text{hor}} + H_I\bar\Theta^I = H_{\text{hor}} + \mathbf{a}(\boldsymbol{F}_I^{2,0} + \boldsymbol{F}_I^{0,2} - \boldsymbol{F}_I^{1,1})\boldsymbol{\Theta}^I$$

  ▷ $\mathcal{N}=2$ solution: $H = H_{\text{hor}} - \frac{\mathbf{a}}{2}(\Pi_0^{1,1}\boldsymbol{F})\bar\Theta - \frac{\mathbf{a}}{2}(\Pi_0^{1,1}\overline{\boldsymbol{F}})\Theta$

- Quantisation of $\mathbf{a}$ ⇔ disconnection between $\mathcal{N}=2$ and $\mathcal{N}=1$ flux vacua:
  ▷ C.S. deformation of $\boldsymbol{F} = F \in H^{1,1}(M)$ can generate $F' \in H^{2,0}(M)$
  ▷ can relate $\mathcal{N}=2$ and $\mathcal{N}=1$ backgrounds via deformation?!?!
  ▷ cf
$$\begin{cases} H_{\text{def}} = (H_{\text{hor}})_{\text{def}} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\boldsymbol{F}+\Pi_s^{2,0}\boldsymbol{F})\bar\Theta - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\overline{\boldsymbol{F}}+\Pi_s^{0,2}\overline{\boldsymbol{F}})\Theta \\ H_{\mathcal{N}=1} = H_{\text{hor}} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\boldsymbol{F}-\Pi_s^{2,0}\boldsymbol{F})\bar\Theta - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\overline{\boldsymbol{F}}-\Pi_s^{0,2}\overline{\boldsymbol{F}})\Theta \end{cases}$$
  ▷ variation of complex structure of $\mathcal{N}=2$ solution either breaks or preserves **ALL** supersymmetry

- The area of $\mathbb{T}^2$ $\mathbf{a}$ is quantised in units of α'
  ▷ resolves 2 problems... need to turn to Bianchi Identity

Heterotic BI and connections with torsion

Fu and Yau showed existence of solutions to susy equations with H flux and BI:

$$dH_3 = 2i\partial\bar{\partial}\omega_X = \frac{\alpha'}{4}(\text{tr}\, R^2(\boldsymbol{\Sigma}_{\text{Chern}}) - \text{tr}\, F^2)$$

▷ Chern (canonical Hermitean) connection on complex manifolds:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma^{\mu}_{\nu} & 0 \\ 0 & \bar{\Sigma}^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix} = \begin{pmatrix} \mathrm{d}z^{\lambda} g_{\nu\bar{\lambda},\lambda} g^{\bar{\lambda}\mu} & 0 \\ 0 & \mathrm{d}\bar{z}^{\bar{\lambda}} g_{\lambda\bar{\nu},\bar{\lambda}} g^{\bar{\mu}\lambda} \end{pmatrix}$$

⊕! When connection is Chern, both sides of the BI are $(2,2)$ forms

▷ Curvature two-form for the connection with torsion ($\Omega_+ = \boldsymbol{\Sigma} + \boldsymbol{T}$):

$$R_+ = \begin{pmatrix} \bar{\partial}\Sigma - \bar{T}T & \bar{\partial}T - \bar{\Sigma}T \\ \partial\bar{T} - \Sigma\bar{T} & \partial\bar{\Sigma} - T\bar{T} \end{pmatrix} + \begin{pmatrix} 0 & \partial T - T\Sigma \\ \bar{\partial}\bar{T} - \bar{\Sigma}\bar{T} & 0 \end{pmatrix} = R_{(1,1)} + R_{(2,0)} + R_{(0,2)}$$

⊖! $R_{(2,0)} \neq 0$ for $F' \neq 0$ ($F' = \boldsymbol{F}_{(2,2)}$) problems for $\mathcal{N} = 1$ solution

⊖!! To satisfy EOM $\mathcal{O}(\alpha')$ two-form R needs to satisfy HYM. $R(\boldsymbol{\Sigma})$ does not!

$\mathbb{T}^2$ area $\mathbf{a}$ is quantised in units of α'

- Bianchi Identity

$$2i\partial\bar{\partial}e^{2\phi}\omega_0 + \mathbf{a}\partial\bar{F}'\Theta + \mathbf{a}\bar{\partial}F'\bar{\Theta} = \frac{\alpha'}{4}\left[\text{tr}\, R_+^2 - \text{tr}\,\mathcal{F}^2 + \frac{4\mathbf{a}}{\alpha'}(F\bar{F} - F'\bar{F}')\right] + \mathcal{O}(\alpha'^2)$$

  ▷ For $\mathbf{a} \sim \alpha'$ all solved by

$$\partial\bar{F}' = 0 \qquad \Rightarrow \qquad \begin{cases} \bullet \quad F' = \lambda\Omega_0 \text{ for } \lambda = \text{const} \\ \bullet \quad \partial F = 0 \end{cases}$$

  ▷ $\phi = \alpha' f/4 \quad \Rightarrow \quad i\partial\bar{\partial}f = \frac{1}{4}\left[\text{tr}\, R_+^2 - \text{tr}\,\mathcal{F}^2 + \frac{4\mathbf{a}}{\alpha'}(F\bar{F} - F'\bar{F}')\right] + \mathcal{O}(\alpha')$

  ▷ In $\mathcal{N} = 2$ case, Ω_+ is *horizontal*. No α' expansion of dilaton is needed. Direct map to Fu-Yau solution (higher α' vanish for $\mathcal{N} = 2$?!)

- HYM for R_+

  ▷ $R_{(2,0)} \sim \mathcal{O}(\alpha')$ provided $\partial F = 0$

  ▷ Can show $\quad \omega_X^2 \wedge R_+^{1,1} = \frac{i\mathbf{a}}{2}\Theta \wedge \bar{\Theta} \wedge \omega_0 \wedge R_+^{1,1} = \mathcal{O}(\alpha'^2)$

Generalise to dual pairs incl. non-geometric backgrounds. 3d theories without 4d lift...

Some open questions:

- Tests of 10 and 11-d couplings:
  - ▷ fixing the ambiguities
  - ▷ higher orders in α'
  - ▷ susy transformations
  - ▷ **LBT** $\Rightarrow$ general formalism for susy theories (with α') corrections (?)
- Lower dimensions and less supersymmetry:
  - ▷ recent progress in construction of four-dimensional $\mathcal{N} = 2$ higher-derivative terms
  - ▷ Implications for consistency (swampland)
  - ▷ subleading terms in AdS/CFT
- Construction of generic $\mathcal{N} = 1$ heterotic flux backgrounds

⋆ Can generalised geometry capture the systematics of the string (perturbation) theory?

⋆ More news from old dualities?