Soft Theorems: trees, loops and strings

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with

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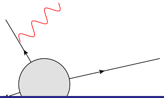


Foreword: Scattering Amplitudes ... see *e.g.*[Elvang, Huang]

- Spinor helicity formalism and SUSY [Grisaru, Pendelton; Van Nieuwenhuizen]
- Color Ordering and dual diagrams [Parke-Taylor; Mangano-Parke; Berends, Giele; Kleis, Kuijf]
- On-shell recursion relations [Britto, Cachazo, Feng, Witten]
- MHV expansion [Cachazo, Svrcek, Witten], Twistor Strings [Witten, Berkovits]
- NMHV and beyond, Momentum Twistors [Hodges; Boels, Mason, Skinner], all-loop formulae [Arkani-H, Bourjaily, Cachazo, Caron-Huot, Trnka]
- (Generalized) Unitary methods [Bern, Dixon, Dunbar, Kosower; Elvang, Freedman, Kiermeier]
- Gravity = $(\text{Gauge})^2$ [Kawai, Lewellen, Tye; Bern, Dixon, Perelstein, Rozowsky; MB, Elvang, Freedman]

Foreword: Scattering Amplitudes ... and more

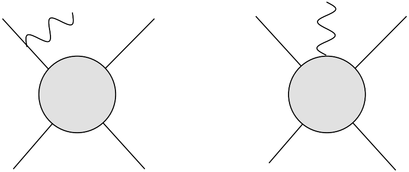
- Amplitude - Wilson loop duality [Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; Eden, Beisert, Staudacher; Bern, Dixon, Smirnov; Broedel; Plefka; . . . Wen]
- Amplitude - Wilson loop - Correlator triality Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; Wen; MB]
- Colour-kinematics duality and ‘double copy’ [Bern, Carrasco, Johanssen]
- Polylogs, . . . symbols [Goncharov; Bern, Del Duca, Schmidt; Spradlin, Volovich; . . .]
- Leading singularities, Grassmannian [Arkani-H, Bourjaily, Cachazo, Trnka; . . .] Bipartite FT [Franco, Galloni, Mariotti, Seong]
- Scattering Equations [Cachazo, He, Yuan; Dolan, Goddard]
- Soft Theorems and BMSV group [Bondi, Metzner, Sachs, Van der Burg; Barnich; Strominger; Casali; Cachazo . . . Schwab, Volovich] . . . Loop corrections? Higher-derivative corrections? Strings?

Plan

- Soft Theorems at tree level: leading, sub-leading, sub-sub-leading
- Loops see also [He, Huang, Wen 1405.1410; Bern, Davies, Nohle]
  - Soft theorems for *Integrands* in $\mathcal{N} = 4$ and in $\mathcal{N} < 4$
  - Finite loop amplitudes in gravity and gauge theories
  - Conformal anomaly and integrated soft theorems
- Higher-derivative interactions
  - F^3
  - R^3 and $R^2\phi$
- Strings
  - Explicit results for $n = 4, 5, 6$ open and closed
  - World-sheet analysis on the disk and the sphere
- Conclusions and Outlook

Soft theorems at tree level

Soft behavior $k_s \to \delta k_s$, $\delta \to 0$ from Ward identity [Low]



(a) (b)

Leading term δ^{-1} from (a) : $\sum_i e_i \frac{\epsilon_s \cdot k_i}{\delta k_s \cdot k_i} A_{n-1}(\ldots, k_i + \delta k_s, \ldots)$

Sub-leading term from (a) and (b): universal operator acting on A_{n-1}

QED [Low]

$$A_n = \delta^{-1} S^{(0)} A_{n-1} + S^{(1)} A_{n-1} + \mathcal{O}(\delta^1)$$

Gravity [Weinberg, Gross-Jackiw]

$$M_n = \delta^{-1} S^{(0)} M_{n-1} + S^{(1)} M_{n-1} + \delta S^{(2)} M_{n-1} + \mathcal{O}(\delta^2)$$

Soft operators at tree level

Gauge theory (color ordered) $A_n = \delta^{-1} S^{(0)}_{YM} A_{n-1} + S^{(1)}_{YM} A_{n-1} + \mathcal{O}(\delta^1)$
where

$$S^{(0)}_{SYM} = \frac{\epsilon_s \cdot k_{s+1}}{k_s \cdot k_{s+1}} - \frac{\epsilon_s \cdot k_{s-1}}{k_s \cdot k_{s-1}}$$

$$S^{(1)}_{SYM} = (k^\mu_s \epsilon^\nu_s - k^\nu_s \epsilon^\mu_s) \left[\frac{J^{s+1}_{\mu\nu}}{k_s \cdot k_{s+1}} - \frac{J^{s-1}_{\mu\nu}}{k_s \cdot k_{s-1}} \right]$$

Gravity $M_n = \delta^{-1} S^{(0)}_G M_{n-1} + S^{(1)}_G M_{n-1} + \delta S^{(2)}_G M_{n-1} + \mathcal{O}(\delta^2)$ where

$$S^{(0)}_{Grav} = \sum_{i \neq s} \frac{k^\mu_i E^s_{\mu\nu} k^\nu_i}{k_s \cdot k_i}$$

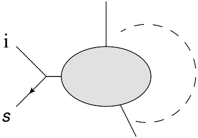
$$S^{(1)}_{Grav} = \sum_{i \neq s} \frac{k^\mu_i E^s_{\mu\nu} J^{\nu\lambda}_i k_{s\lambda}}{k_s \cdot k_i}$$

$$S^{(2)}_{Grav} = \sum_{i \neq s} \frac{k_s \cdot J_i \cdot E_s \cdot J_i \cdot k_s}{k_s \cdot k_i}$$

Holomorphic soft limit from BCFW

Tree-level (holomorphic) soft limit via recursion relations, helicity spinors ($k^2 = 0 \leftrightarrow k_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$ in $D = 4$) [Strominger, Cachazo; Casali]

$$\lambda_s \to \delta\lambda_s + z\lambda_n \quad , \quad \lambda_n \to \lambda_n - z\delta\lambda_s$$



$$+ \quad \ldots\ldots = \sum_{1\leq i<n} M_3(\hat s^+, i, -\hat K_{is}) \frac{1}{K^2_{i,s}} M_n(\hat K_{is}, \ldots, \hat n) + R^{fin}_{mass}$$

$$M_3(\hat s^+, i, \hat K_{is}) \frac{1}{K^2_{is}} M_n(-\hat K_{is}, \ldots, \hat n) = \frac{1}{\delta^3} S^0_{s,i} M_n \left(\tilde\lambda_i + \delta \frac{\langle s\, n\rangle}{\langle i\, n\rangle} \tilde\lambda_s, \tilde\lambda_n + \delta \frac{\langle s\, i\rangle}{\langle n\, i\rangle} \tilde\lambda_s \right)$$

with inverse soft function $S^{(0)}_{s,i} = (\langle ni\rangle^2/\langle ns\rangle^2)([is]/\langle is\rangle)$ independent of helicity h_i. Exponentiation [S.He, Y-t Huang, C. Wen 1405.1410]

$$= \delta^{-3}\, S^{(0)}_{s,i}\ \exp \delta \left(\frac{\langle s\, n\rangle}{\langle i\, n\rangle} \tilde\lambda_s \cdot \frac{\partial}{\partial \tilde\lambda_i} + \frac{\langle s\, i\rangle}{\langle n\, i\rangle} \tilde\lambda_s \cdot \frac{\partial}{\partial \tilde\lambda_n} \right) M_n$$

Further developments [S.He, Y-t Huang, C. Wen 1405.1410]

Using helicity spinors in $D = 4$

- Soft theorems for MHV
- Double Copy

$$S^{(k)}_{\mathrm{Grav}} = \sum_i K_{is} [S^{(0)}_{\mathrm{YM}}(i) e^{J/2}]^2$$

- Supersymmetric version:

$$\mathcal{J} = \left[\frac{\langle sn \rangle}{\langle in \rangle} \left(\tilde{\lambda}_s \cdot \frac{\partial}{\partial \tilde{\lambda}_i} + \eta_s \cdot \frac{\partial}{\partial \eta_i} \right) + \frac{\langle si \rangle}{\langle ni \rangle} \left(\tilde{\lambda}_s \cdot \frac{\partial}{\partial \tilde{\lambda}_n} + \eta_s \cdot \frac{\partial}{\partial \eta_n} \right) \right]$$

- Sub-leading YM soft operator ($s = n + 1$)

$$S^{(1)}_{\mathrm{YM}} = \frac{p_s^{\alpha\dot{\alpha}}}{\langle ns \rangle \langle s1 \rangle [1n]} \frac{1}{2} \left[p_{n\alpha\dot{\beta}} (M_1{}^{\dot{\beta}}{}_{\dot{\alpha}} - \delta^{\dot{\beta}}{}_{\dot{\alpha}} (D_1 - h_1)) - (n \leftrightarrow 1) \right] ,$$

Kinematic simplification

Kinematics and holomorphic shift in the soft limit $\tilde{\lambda}_s \to \delta \tilde{\lambda}_s$

$$A_{n+1} = \left(S^{(0)}_{YM} + \delta S^{(1)}_{YM} \right) A_n + \mathcal{O}(\delta^2)$$

$$S^{(1)}_{YM} = S^{(0)}_{YM} \left[\frac{\langle sn \rangle}{\langle in \rangle} \tilde{\lambda}_s \cdot \frac{\partial}{\partial \tilde{\lambda}_1} + \frac{\langle s1 \rangle}{\langle n1 \rangle} \tilde{\lambda}_s \cdot \frac{\partial}{\partial \tilde{\lambda}_n} \right]$$

Solve momentum conservation $\sum_i p_i = 0$ with

$$|1] = - \sum_{i=2}^{n-1} \frac{\langle ni \rangle [i|}{\langle n1 \rangle}, \quad |n] = - \sum_{i=2}^{n-1} \frac{\langle 1i \rangle [i|}{\langle 1n \rangle},$$

Sub-leading term vanishes $S^{(1)}_{YM} A_n = 0$

$$A_{n+1} = S^{(0)}_{YM} A_n + \mathcal{O}(\delta^2)$$

1. What happens at loop level ?

- IR-divergences [Bern, Davies, Nohle]
- IR-finite rational terms [Bern, Davies, Nohle; S. He, C. Wen, Y-t Huang]

$$\begin{aligned} A_{n+1}(+ + + \ldots +) &\rightarrow A_n(+ + \ldots +) \quad \text{(Exact)} \\ A_{n+1}(- + + \ldots +) &\rightarrow A_n(+ + \ldots +) \quad \text{(Exact)} \\ A_{n+1}(- + + \ldots +) &\rightarrow A_n(- + \ldots +) \quad \text{(Corrected)} \\ \Delta S^{(1)} A_n(- + \ldots +) &= -S^{(0)} \frac{[n\, n{+}1]}{\langle n\, n{+}1\rangle} A_n^{(0)}(1^-, 2^+, \ldots, (n-1)^+, n^-) \end{aligned}$$

For pure gravity M_n, Soft operators corrected at loop-level

- Soft theorems for integrands? Do $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$ commute?

[Cachazo, Yuan]

$$\int d\ell^{4-2\epsilon} \frac{1}{(\ell + \delta s_{si})^2} \qquad \ell \quad \text{hard wrt} \quad \delta k_s\ ?$$

Manifest soft integrands

Is there a representation of loop *integrands* well-behaved in the soft limit?

$$\mathrm{I}_{\mathrm{n}+1} = \mathrm{S}^{(0)}_{\mathrm{YM}} \mathrm{I}_{\mathrm{n}} + \mathcal{O}(\delta^2)$$

Use super momentum twistors [Hodges; Boels, Mason, Skinner]

$$\mathcal{Z} = \{Z^I | \chi^A\} = \{\lambda^\alpha, \mu^{\dot\alpha} | \chi^A\}$$

to trivialize (super) momentum conservation

$$\tilde\lambda_{\mathrm{a}} = \frac{\mu_{\mathrm{a}-1}}{\langle \mathrm{a}, \mathrm{a}+1\rangle} + \frac{\langle \mathrm{a}-1, \mathrm{a}+1\rangle \mu_{\mathrm{a}}}{\langle \mathrm{a}-1, \mathrm{a}\rangle\langle \mathrm{a}, \mathrm{a}+1\rangle} + \frac{\mu_{\mathrm{a}+1}}{\langle \mathrm{a}-1, \mathrm{a}\rangle}$$

same for η_a^A and χ_a^A $A = 1, \dots \mathcal{N}$ [Drummond, Henn]

Anti-holomorphic soft limit: $\mathrm{Z}_{\mathrm{n}} \to \alpha \mathrm{Z}_1 + \beta \mathrm{Z}_{\mathrm{n}-1} + \delta \mathrm{Z}_{\mathrm{s}}$, since

$$\tilde\lambda_{\mathrm{n}} = \delta\, \frac{\langle \mathrm{n}-1, 1\rangle \mu_{\mathrm{s}} + \langle 1, \mathrm{s}\rangle \mu_{\mathrm{n}-1} + \langle \mathrm{s}, \mathrm{n}-1\rangle \mu_1}{\langle 1, \mathrm{n}-1\rangle^2 \alpha\beta} = \delta \hat{\tilde\lambda}_s$$

Planar $\mathcal{N} = 4$ SYM

$$A_{n,k}^{(L)} = A_0^{MHV} \int d^D\ell_1 \dots d^D\ell_L R_{n,k}^{(L)} \quad \text{with} \quad A_0^{MHV} = \frac{\delta^{4|2\mathcal{N}}(\lambda_a\tilde{\lambda}_a|\eta_a^A\tilde{\lambda}_a)}{\langle 1,2\rangle \dots \langle n,1\rangle}$$

All-loop recursion relation [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]

$$\begin{aligned} R_{n,k}^{(L)} &= R_{n-1,k}^{(L)} + \int_{GL(2)} [1,A,B,n-1,n] R_{n+2,k+1}^{(L-1)}(1,\dots,\hat{n},A,\hat{B}) \\ &+ \sum_{L',k',i} R_{i,k'}^{(L')}(1,\dots,i-1,I_i)[1,i-1,i,n-1,n] R_{n+2-i,k-1-k'}^{(L-L')}(I_i,i,\dots,\hat{n}_i) \end{aligned}$$

where $\hat{n}_i = (n-1n)\cap(1i-1i)$, $I_i = (i-1i)\cap(1n-1n)$,
$\hat{n} = (n-1n)\cap(1AB)$, $\hat{B} = (AB)\cap(1n-1n)$
with 'overlap': $(ab)\cap(ijk) = \mathcal{Z}_a\langle bijk\rangle - \mathcal{Z}_b\langle aijk\rangle$
4-pt invariant: $\langle abcd\rangle = \epsilon_{IJKL}Z_a^I Z_b^J Z_c^K Z_d^L$ and 5-pt R-invariant:

$$[a,b,c,d,e] \equiv \frac{\delta^{0|4}(\chi_a^A\langle bcde\rangle + cyc)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}.$$

Planar $\mathcal{N} = 4$ SYM: Soft limit (I)

$$Z_n \to \alpha Z_{n-1} + \beta Z_1 + \delta Z_s$$

First consider 'factorisation' terms

$$\begin{aligned} R^{(L)}_{n,k} &= R^{(L)}_{n-1,k} \\ &+ \sum_{L',k',i} R^{(L')}_{i,k'}(1,\ldots,i-1,I_i)[1,i-1,i,n-1,n]R^{(L-L')}_{n+2-i,k-1-k'}(I_i,i,\ldots,\hat{n}_i) \\ &\quad I_i = \delta(i-1i)\cap(1n-1s), \quad \mathcal{Z}_{\hat{n}_i} = \mathcal{Z}_1 + \mathcal{O}(\delta) \\ &+ \int_{GL(2)} [1,A,B,n-1,n]R^{(L-1)}_{n+2,k+1}(1,\ldots,\hat{n},A,\hat{B}) \end{aligned}$$

$$[1,i-1,i,n-1,n] \to \frac{\delta^4 \times \delta^{0|4}(\chi_{[i-1}\langle i],n-1,s,1\rangle + \chi_s\langle 1,i-1,i,n\rangle)}{\alpha\,\beta\,\delta^2\langle 1,i-1,i,n-1\rangle^3\langle n-1,s,1,i-1\rangle\langle n-1,s,1,i\rangle} + \mathcal{O}(\delta^3)$$

Planar $\mathcal{N} = 4$ SYM: Soft limit (II)

$$Z_n \to \alpha Z_{n-1} + \beta Z_1 + \delta Z_s$$

Then consider 'forward-limit' term

$$\begin{aligned}
R^{(L)}_{n,k} &= R^{(L)}_{n-1,k} \\
&+ \sum_{L',k',i} R^{(L')}_{i,k'}(1,\ldots,i-1,I_i)[1,i-1,i,n-1,n]R^{(L-L')}_{n+2-i,k-1-k'}(I_i,i,\ldots,\hat{n}_i) \\
&\mathcal{O}(\delta^2) \\
&+ \int_{GL(2)} [1,A,B,n-1,n]R^{(L-1)}_{n+2,k+1}(1,\ldots,\hat{n},A,\hat{B}) \\
&\mathcal{Z}_{\hat{n}} = \mathcal{Z}_1 + \mathcal{O}(\delta), \quad \mathcal{Z}_{\hat{B}} = \delta(AB)\cap(1n-1s)
\end{aligned}$$

where $[1,A,B,n-1,n] \to$

$$\frac{\delta^2}{\alpha\beta}\frac{\delta^{0|4}(\chi_{[A}\langle B],n-1,s,1\rangle + \chi_s\langle 1,A,B,n\rangle)}{\langle 1,A,B,n-1\rangle\langle A,B,n-1,1\rangle\langle 1,A,n-1,s\rangle\langle 1,B,n-1,s\rangle\langle n-1,1,A,B\rangle}$$

Planar $\mathcal{N} = 4$ SYM: Soft limit (III)

To any loop order in planar $\mathcal{N} = 4$ SYM

$$\mathrm{R}^{(\mathrm{L})}_{\mathrm{n,k}} \to \mathrm{R}^{(\mathrm{L})}_{\mathrm{n-1,k}} + \mathcal{O}(\delta^2)$$

then

$$\mathrm{I}_{\mathrm{n+1}} = \left(\mathrm{S}^{(0)}_{\mathrm{YM}} + \delta \mathrm{S}^{(1)}_{\mathrm{YM}}\right) \mathrm{I}_{\mathrm{n}} + \mathcal{O}(\delta^2)$$

Universal soft behaviour at all loops at *integrand* level!

Integrands with manifest soft behavior have interesting properties
. . . not explicit . . . soft guidance as how to find them?

$\mathcal{N} < 4$ SYM

SYM theories with lower susy

$$\mathcal{N} < 4 \text{ SYM} = \mathcal{N} = 4 \text{ SYM} - (4 - \mathcal{N}) \;\; \text{Chiral}$$

Use CSW expansion [Brandhuber, Spence, Travaglini]

$$\mathrm{R}^{(1),\mathrm{chiral}}_{\mathrm{n},2} = \mathrm{R}^{(1),\mathrm{chiral}}_{\mathrm{n}-1,2}$$

$$+ \quad \frac{\langle \mathrm{a}\hat{\mathrm{B}}\rangle\langle \mathrm{b}\hat{\mathrm{B}}\rangle}{\langle \mathrm{AB}\rangle^2 \langle \mathrm{AB1n{-}1}\rangle\langle \mathrm{ABn{-}1n}\rangle\langle \mathrm{AB1n}\rangle} \sum_{\mathrm{a}<\mathrm{i}\leq \mathrm{b}} \frac{\langle \mathrm{aI_i}\rangle\langle \mathrm{bI_i}\rangle}{\langle \mathrm{AB1i{-}1}\rangle\langle \mathrm{ABi{-}1i}\rangle\langle \mathrm{AB1i}\rangle}$$

$$\mathcal{Z}_{\hat{B}} = \delta(AB) \cap (1n{-}1s)$$

Thus

$$\mathrm{R}^{(1)}_{\mathrm{n},2} \to \mathrm{R}^{(1)}_{\mathrm{n}-1,2} + \mathcal{O}(\delta^2)$$

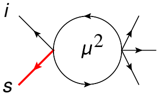
Soft theorems are manifest for planar integrands in $\mathrm{D} = 4 - 2\epsilon$ dimensions

All Plus Rational terms

Why do the two limits commute for all-plus*? [Bern, Morgan]

$$A_5^{YM}(+,+,+,+,+) = \frac{2}{\prod_{i=1}^{5}\langle i i+1\rangle}\left(-\frac{1}{2}\left[\frac{\mu^4 s_{12}s_{23}}{d_1 d_2 d_3 d_5} + \text{cyclic}\right] + \frac{4i\mu^6\epsilon(1234)}{d_1 d_2 d_3 d_4 d_5}\right)$$

$$I_m[\mu^{2r}] = -\epsilon(1-\epsilon)\dots(r-1-\epsilon)(4\pi)^r I_m^{D=4+2r-2\epsilon}\,,$$



Dimension shifting: [Bern, Dixon, Perelstein, Rozowsky]

$$R_n^{YM}(+,\dots,+) = \mu^4 R_n(\mathcal{N}=4\ \text{SYM})$$
$$R_n^{Grav}(+,\dots,+) = \mu^8 R_n(\mathcal{N}=8\ \text{Sugra})$$

Non-commutativity stems from scalar bubbles

* In susy theories $A_n(+,+,\dots+) = 0 = A_n(-,+,\dots+)$

Soft corrections from anomalies

$$A_{n+1}(\{\lambda_s, \tilde{\lambda}_s\}, \delta\lambda_s, \tilde{\lambda}_s) = \frac{1}{\delta^2} S^{(0)} A_n + \frac{1}{\delta} S^{(1)} A_n$$

Under conformal boosts

$$\mathfrak{K} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i \partial\tilde{\lambda}_i} + \frac{\partial^2}{\delta\partial\lambda_s\partial\tilde{\lambda}_s} = \mathfrak{K}_0 + \frac{1}{\delta}\mathfrak{K}_s$$

neglecting anomalies $\mathfrak{K}A_{n+1} = 0$: conformal invariance implies a differential equation for soft functions [Larkoski]

$$\begin{aligned} S^{(0)} &= \frac{\langle 1n\rangle}{\langle 1s\rangle\langle sn\rangle} \\ \mathcal{O}(\delta^{-2}):\quad \mathfrak{K}_0 S^{(0)} A_n + \mathfrak{K}_s S^{(1)} A_n &= -\left(\frac{\lambda_n}{\langle ns\rangle^2}\frac{\partial}{\partial\tilde{\lambda}_n} + \frac{\lambda_1}{\langle 1s\rangle^2}\frac{\partial}{\partial\tilde{\lambda}_1}\right) A_n \\ &+ (\mathfrak{K}_s S^{(1)}) A_n = 0 \end{aligned}$$

Conformal anomalies

- Tree-level soft functions are homogenous solutions to CDE
- Loop level corrections must arise from conformal anomalies

$$(\mathfrak{K}_0 + \frac{1}{\delta}\mathfrak{K}_s)A_{n+1}(\{\lambda_i, \tilde{\lambda}_i\}, \delta\lambda_s, \tilde{\lambda}_s) = \sum_r a^{(r)}_{n+1}\delta^r$$

- All-plus amplitude $\mathrm{A_n}(+,+,\ldots,+)$ conf-invariant (Self-dual)
- Single-minus amplitude $\mathrm{A_n}(-,+,\ldots,+)$ not conformally invariant, only scale invariant. Depending on $h = \pm$ of 'soft' leg

$$\mathrm{A_{n+1}}(- + + \ldots +) \to a^{(0)}_{n+1} + \ldots \quad \mathrm{A_{n+1}}(- + + \ldots +) \to \frac{1}{\delta^2} a^{(-2)}_{n+1} + \ldots$$

$$\mathcal{O}(\delta^{-2}) \qquad \mathfrak{K}_0 S^{(0)} A_n + \mathfrak{K}_s(S^{(1)} + \Delta^{(1)})A_n = a^{(-2)}_{n+1}\,.$$

$$\Delta^{(1)}\mathrm{A_n} = -\tilde{\lambda}^{\dot{a}}_s \lambda^{a}_s (\mathrm{S}^{\mathrm{tree}(0)}\mathfrak{K}_0 \mathrm{A_n} - \mathrm{a}^{(-2)}_{\mathrm{n}+1})\,.$$

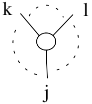
Universality would imply $a^{(-2)}_{n+1} = \mathcal{O}A_n$

2. Are soft theorems universal ?

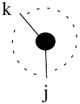
- Effective field theories with F^3, R^3 (Forbidden by susy, appear in bosonic strings)
- Effective field theories with $R^2\phi$ (appears in SUGRA due to U(1) anomalies and in bosonic strings) [Carrasco, Kallosh, Roiban, Tseytlin]
- String theory at finite α' [MB, He, Huang, Wen; Bern, Di Vecchia, Nohle; ...]

F^3 coupling

CSW representation, two ‘basic’ vertices [Broedel, Dixon]

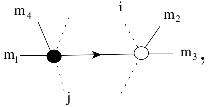


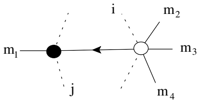
: $\dfrac{\langle jk\rangle^2\langle kl\rangle^2\langle lj\rangle^2}{\prod_{i=1}^{n}\langle i i+1\rangle}$ (new F^3_{SD})



: $\dfrac{\langle jk\rangle^4}{\prod_{i=1}^{n}\langle i i+1\rangle}$ (usual MHV)

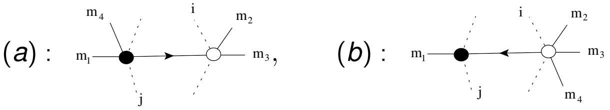
Consider diagrams with only one white vertex

(a) : 

(b) : 

F^3 amplitudes

CSW-like expansion with two 'MHV' vertices



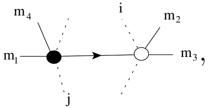
$$(a): \quad \frac{1}{\prod_{l=1}^{n}\langle l\, l{+}1\rangle}\langle m_1 m_4\rangle^4\ [i\, i{-}1\, j\, j{-}1\, *]\ \langle m_2 m_3\rangle^2\langle m_3 \widehat{i{-}1}\rangle^2\langle \widehat{i{-}1} m_2\rangle^2$$

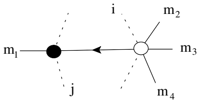
$$(b): \quad \frac{1}{\prod_{l=1}^{n}\langle l\, l{+}1\rangle}\langle m_1 \widehat{i{-}1}\rangle^4\ [i\, i{-}1\, j\, j{-}1\, *]\ \langle m_2 m_3\rangle^2\langle m_3 m_4\rangle^2\langle m_4 m_2\rangle^2\,,$$

where $*$ denotes reference twistor $Z_* = (0,\mu,0)$ and $\widehat{i{-}1} = (i\,i-1)\cup(*\,j\,j-1)$. For instance

$$R^{F^3}_{1,n} = [i\,i{-}1\,j\,j{-}1\,*](\langle m_1 m_4\rangle^4\langle m_2 m_3\rangle^2\langle m_3 \widehat{i{-}1}\rangle^2\langle \widehat{i{-}1}\, m_2\rangle^2 + \ldots)$$

Soft limit of F^3 amplitudes

(a) : 

(b) : 

$$(a): \quad \frac{1}{\prod_{l=1}^{n}\langle l\, l+1\rangle}\langle m_1 m_4\rangle^4\ [i\, i-1\, j\, j-1*]\ \langle m_2 m_3\rangle^2 \langle m_3 \widehat{i-1}\rangle^2 \langle \widehat{i-1} m_2\rangle^2$$

$$(b): \quad \frac{1}{\prod_{l=1}^{n}\langle l\, l+1\rangle}\langle m_1 \widehat{i-1}\rangle^4\ [i\, i-1\, j\, j-1*]\ \langle m_2 m_3\rangle^2 \langle m_3 m_4\rangle^2 \langle m_4 m_2\rangle^2\,,$$

Recursion relation à la BCFW and soft limit

$$R^{F^3}_{k,\,n} = R^{F^3}_{k,\,n-1} + \sum_j [n-1, n, 1, j-1, j] R^{F^3}_{k',\,j} R^{F^2}_{k-1-k',\,n+2-j} + (F^3 \leftrightarrow F^2)$$

$$R^{F^3}_{k,\,n} = R^{F^3}_{k,\,n-1} + \mathcal{O}(\delta^2) \qquad A_{n+1} = S^{(0)}_{YM} A_n + \mathcal{O}(\delta^2)$$

The presence of F^3 does not spoil $S^{(1)}_{F^3} = S^{(1)}_{YM}$

$R^3/R^2\phi$ amplitudes

For gravity, use KLT-like formulae

$$M(1^-,2^-,3^-,4^-,5^+) =$$

$$i s_{12} s_{34} A^{F^3}(1^-,2^-,3^-,4^-,5^+) A^{F^3}(2^-,1^-,4^-,3^-,5^+) + \mathcal{P}(2,3)$$

There is a mixing of operators (α'^2) R^3, (α'^1) ϕR^2

$$M_{n+1} \to \frac{1}{\delta^3} S^{(0)} M_n + \frac{1}{\delta^2} S^{(1)} M_n + \frac{1}{\delta} S^{(2)} M_n + \Delta^{(2)} + \mathcal{O}(\delta^0)$$

$$\Delta^{(2)} = \sum_j -2 \frac{\langle 1j \rangle^3}{[1j]} M_n(\phi, i_1^-, i_2^-, \ldots, i_{n-2}^-, n^+)$$

For $\mathcal{N} \leq 4$ there are U(1) anomalies: $R^2\phi$ generated at one-loop

[Carrasco, Kallosh, Roiban, Tseytlin]

Same for bosonic string already at tree level

Open superstring amplitudes on the disk

Use KLT-like representation [Mafra, Schlotterer, Stieberger]

$$A_n = \sum_{\sigma \in S_{n-3}} F^{(2_\sigma,\ldots,(n-2)_\sigma)} A_{YM}(1, 2_\sigma, \ldots, (n-2)_\sigma, n-1, n)$$

For $n = 4$, $(n-3)! = 1$, setting $s_{ij} = 2\alpha' k_i \cdot k_j$

$$A_4 = F^{(2)} A_{YM} = s_{12} \int_0^1 dz_2\, z_2^{s_{12}-1} (1 - z_2)^{s_{23}} A_{YM}$$

For $n = 5$, $(n-3)! = 2$

$$A_5 = F^{(2,3)} A_{YM}(1,2,3,4,5) + F^{(3,2)} A_{YM}(1,3,2,4,5)$$

$$\begin{aligned} F^{(2,3)} &= s_{12}s_{34} \int_0^1 dz_2 \int_{z_2}^1 dz_3 z_2^{s_{12}-1} z_3^{s_{13}} z_{32}^{s_{23}} (1 - z_2)^{s_{24}} (1 - z_3)^{s_{34}-1}\,, \\ F^{(3,2)} &= s_{13}s_{24} \int_0^1 dz_2 \int_{z_2}^1 dz_3 z_2^{s_{12}} z_3^{s_{13}-1} z_{32}^{s_{23}} (1 - z_2)^{s_{24}-1} (1 - z_3)^{s_{34}} \end{aligned}$$

Soft limit of 5-pt amplitude

Integrate over z_3 and keep terms up to sub-leading order in δ

$$F^{(2,3)} \quad \rightarrow \quad s_{12} \int_0^1 dz_2\, z_2^{s_{12}-1}(1-z_2)^{s'_{24}} \left[1 + \delta(s_{23} + s'_{34} + k_2 \cdot p_4) \log(1-z_2)\right].$$

Fix $D = 4$ kinematics:

$$k'_4 = \frac{|4\rangle\langle 5|(1+2)}{\langle 45\rangle},\ p_4 = \frac{|4\rangle\langle 5|3}{\langle 45\rangle},\ k'_5 = \frac{|5\rangle\langle 4|(1+2)}{\langle 54\rangle},\ p_5 = \frac{|5\rangle\langle 4|3}{\langle 54\rangle}$$

$$F^{(2,3)}_{S^{(1)}} \quad = \quad \frac{\langle 34\rangle\langle 51\rangle[31]}{\langle 45\rangle} s_{12} \int_0^1 dz_2\, z_2^{s_{12}-1}(1-z_2)^{s'_{24}} \log(1-z_2)$$

$$F^{(3,2)}_{S^{(1)}} \quad = \quad -s_{13}s'_{24} \int_0^1 dz_2 z_2^{s_{12}}(1-z_2)^{s'_{24}-1} \log(z_2)$$

$$S^{(1)}_{YM}(234)A(1,2,4,5) = \left(\frac{\tilde\lambda_3}{\langle 23\rangle}\cdot\frac{\partial}{\partial\tilde\lambda_2} + \frac{\tilde\lambda_3}{\langle 34\rangle}\cdot\frac{\partial}{\partial\tilde\lambda_4}\right) A(1,2,4,5)$$

Soft expansion is valid for finite α'. Explicit check for 6-pt amplitude

Closed superstring amplitudes

Use (generalized) KLT formulae [Kawai, Lewellen, Tye]

$$
\begin{aligned}
\mathrm{M_n} \quad &= \quad \mathcal{A}_n(1,2,\ldots,n)[\sum_{\{i\},\{j\}} f(i_1,\ldots,i_{\lfloor\frac{n}{2}\rfloor-1})\bar{f}(j_1,\ldots,j_{\lfloor\frac{n}{2}\rfloor-2})\times \\
&\times \mathcal{A}_n(\{i\},1,n-1,\{j\},n) + \mathrm{Perm}(2,\ldots,n-2)]
\end{aligned}
$$

with momentum kernels [Bjerrum-Bohr, Vanhove]

$$
\begin{aligned}
f(i_1,\ldots,i_m) \quad &= \quad \sin(\pi s_{1,i_m}) \prod_{k=1}^{m-1} \sin\pi\left(s_{1,i_k} + \sum_{l=k+1}^{m} g(i_k,i_l)\right), \\
\bar{f}(j_1,\ldots,j_m) \quad &= \quad \sin(\pi s_{j_1,n-1}) \prod_{k=2}^{m} \sin\pi\left(s_{j_k,n-1} + \sum_{l=1}^{k-1} g(j_l,j_k)\right),
\end{aligned}
$$

After some algebra ... get expected soft behaviour $S^{(k)}_{\rm closed} = S^{(k)}_{\rm Grav}$ for $k = 0, 1, 2$

World-sheet analysis

Open superstring amplitudes on the disk

$$A(1,2,\ldots,n) = i g_s^{n-2} \int dz_2 \ldots dz_{n-2} \langle c V(1) V(2) \ldots c V(n-1) c V(n) \rangle$$

Take two vertices in the $q = -1$ picture and the remaining $n - 2$ in the $q = 0$ picture:

$$V_A^{(-1)} = (\epsilon \cdot \psi) e^{-\varphi} e^{ikX} \quad , \quad V_A^{(0)} = (\epsilon \cdot \partial X + ik \cdot \psi \epsilon \cdot \psi) e^{ikX}$$

$V_A^{(0)}$ total derivative as $k \to 0$, only contribution from the two ends

Use OPE

$$V_A^{(0)}(z_s) V_A^{(-1)}(z_{s\pm1}) \approx |z_s - z_{s\pm1}|^{2k_s k_{s\pm1} - 1} e^{-\varphi(z_{s\pm1})} e^{i(k_s + k_{s\pm1}) X(z_{s\pm1})} \times$$

$$\{\epsilon_s \cdot k_{s\pm1} \epsilon_{s\pm1} \cdot \psi + \epsilon_{s\pm1} \cdot k_s \epsilon_s \cdot \psi - \epsilon_{s\pm1} \cdot \epsilon_s \cdot k_s \psi + \epsilon_s \cdot k_{s\pm1} k_s \cdot X_s \epsilon_{s\pm1} \cdot \psi\}(z_{s\pm1}) + \ldots$$

Soft limit of disk amplitudes

Integrate around the singularity $\int_{z_s \approx z} dz_s |z_s - z|^{2k_s \cdot k - 1} = 1/k_s \cdot k$

Combine with first term, as expected, get

$$S^{(0)}_{open} = S^{(0)}_{SYM} = \frac{\epsilon_s \cdot k_{s+1}}{k_s \cdot k_{s+1}} - \frac{\epsilon_s \cdot k_{s-1}}{k_s \cdot k_{s-1}}$$

Next consider sub-leading terms (in k_s), including expansion of $e^{ik_s \cdot X}$

Keep only BRST invariant operators

$$(i[k_s \cdot X \epsilon_s \cdot k_{s\pm1} - \epsilon_s \cdot X k_s \cdot k_{s\pm1}]\epsilon_{s\pm1} \cdot \psi + [\epsilon_{s\pm1} \cdot k_s \epsilon_s \cdot \psi - \epsilon_{s\pm1} \epsilon_s \cdot k_s \cdot \psi])e^{-\varphi} e^{ik_{s\pm1} \cdot X}$$

$$= (k_s^\mu \epsilon_s^\nu - k_s^\nu \epsilon_s^\mu) \left[\tilde{k}_\mu \frac{\partial}{\partial \tilde{k}^\nu} + \tilde{\epsilon}_\mu \frac{\partial}{\partial \tilde{\epsilon}^\nu} \right] V^{(-1)}_{\tilde{A}}(k_{s\pm1})$$

Again

$$S^{(1)}_{open} = S^{(1)}_{SYM} = (k_s^\mu \epsilon_s^\nu - k_s^\nu \epsilon_s^\mu) \left[\frac{J^{s+1}_{\mu\nu}}{k_s \cdot k_{s+1}} - \frac{J^{s-1}_{\mu\nu}}{k_s \cdot k_{s-1}} \right]$$

Closed strings on the sphere

Graviton vertex operators in the $(-1,-1)$ and $(0,0)$ pictures

$$V_G^{(-1,-1)} = E_{\mu\nu}\psi^\mu\tilde{\psi}^\nu e^{-\varphi-\tilde{\varphi}}e^{ik\cdot X}$$

$$V_G^{(0,0)} = E_{\mu\nu}(\partial X^\mu + ik\cdot\psi\psi^\mu)(\bar{\partial}X^\nu + ik\cdot\tilde{\psi}\tilde{\psi}^\nu)e^{ikX}$$

with $E_{\mu\nu} = \epsilon_\mu\tilde{\epsilon}_\nu$ to factorise Left- and Right-movers

Use OPE from open strings for each of the $n-1$ 'hard' gravitons, get

$$S^{(0)}_{\text{closed}} = S^{(0)}_{\text{Grav}} = \sum_{i\neq s}\frac{k_i^\mu E^s_{\mu\nu}k_i^\nu}{k_s\cdot k_i}$$

$$S^{(1)}_{\text{closed}} = S^{(1)}_{\text{Grav}} = \sum_{i\neq s}\frac{k_i^\mu E^s_{\mu\nu}J_i^{\nu\lambda}k_{s\lambda}}{k_s\cdot k_i}$$

$$S^{(2)}_{\text{closed}} = S^{(2)}_{\text{Grav}} = \sum_{i\neq s}\frac{k_s\cdot J_i\cdot E_s\cdot J_i\cdot k_s}{k_s\cdot k_i}$$

Other massless states

- Soft dilaton and soft Kalb-Ramond decouple at leading δ^{-1} order since $\mathrm{k}_\mathrm{i}^2 = 0$ and $\mathrm{k}_\mathrm{i}^\mu \mathrm{k}_\mathrm{i}^\nu \mathrm{B}_{[\mu\nu]} = 0$
- Yet non vanishing sub-leading δ^0 behaviour as for pions [Adler]
- Derivative wrt to 'moduli' fields, (not) puzzling: single soft limit $\rightarrow$ 0, yet 'double' soft limit non-vanishing! [Arkani-H, Cachazo, Kaplan; MB, Elvang, Freedman; Kallosh; ... MB, Consoli, Di Vecchia, Guerrieri, Marotta]

Conclusions and outlook

Conclusions

- Leading and sub-leading soft functions universal (at tree level!)
- Sub-sub-leading corrections are sensitive to scalar couplings $F^2\phi$, $R^2\phi$ (even at tree level!)
- Loops: manifest soft *integrands* exists for planar $\mathcal{N} \leq 4$ SYM
- Loop corrections are intimately tied to anomalies
- Open and closed superstrings same soft behavior as SYM and SuGra at tree level

Outlook

- IR-div corrections are known and physical ...
- The tree-level YM soft functions are homogenous solution to conformal symmetry. Gravity?
- String theory soft theorems simply from OPE. Higher genus?
- Soft 'dilaton/moduli' ....