The Geometry of $N = 2$ Flux Backgrounds

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Joint ERC workshop
"MassTeV, Superfields and Strings & Gravity"
16 October 2013

to appear

The problem

Type II on M_6 with $N = 2$

- what is geometry of generic flux background?
- can one find moduli?

integrable structure in (exceptional) generalized geometry

Calabi–Yau

Moduli spaces well understood

$$\mathrm{d}J = 0 \qquad h^{1,1}$$

$$\mathrm{d}\Omega = 0 \qquad h^{2,1}$$

with harmonic B, $C^{\pm}$

		type IIA	type IIB
hypers	quaternionic Kähler	$\Omega + C^-$	$J + B + C^+$
vectors	special Kähler	$J + B$	Ω

What about generic $N = 2$ flux backgrounds?

- analogues of J and Ω? differential conditions?
- finite dimensional? cohomology?

One approach: G structures

$GL(6,\mathbb{R})$ action on frames $v^\mu = v^a \hat{e}_a^\mu$

$$\mathrm{d}J \simeq \text{flux} \qquad Sp(3,\mathbb{R}) \text{ structure}$$

$$\mathrm{d}\Omega \simeq \text{flux} \qquad SL(3,\mathbb{C}) \text{ structure}$$

- complete classification, new solutions
- lack of integrability means moduli hard; global issues

[*Gauntlett, Martelli, DW,. . . , Cardoso et al,. . .*]

"Generalised" G structure

- extend notion of structure
- new object is integrable

conditions from $N = 2$ gauged supergravity . . .

[c.f. GMTP, Koerber & Martucci, Tomasiello; GLSW, Graña & Orsi, Graña & Triendl]

Gauged hypermultiplets

Multiplets

$$\text{hypers}: (\zeta_\alpha, q^u) \quad \text{four scalars}$$
$$\text{vectors}: (A_\mu, \lambda^A, t) \quad \text{complex scalar}$$

Hypermultiplet manifold (n hypers)

$$M_H \text{ is quaternionic-Kähler manifold}$$

and M_H is $4n$-dimensional

Swann space X: R^4 bundle over M_H

X is hyper-Kähler manifold

with triplet of symplectic structures Ω_a, $a = 1, 2, 3$

Gauging by vector multiplets

action of Lie group G on X preserving Ω_a

isometries of HK metric so $\partial_\mu q^u \to D_\mu q^u$

Moment map

Infinitesimally G acts by Lie derivative $\rho : \mathfrak{g} \to TX$ so for all $\xi \in \mathfrak{g}$

$$0 = \mathcal{L}_{\rho(\xi)}\Omega_a = \mathrm{d}(i_{\rho(\xi)}\Omega_a)$$

Moment maps $\mu_a : X \to \mathfrak{g}^*$ such that

$$\mathrm{d}\mu_a(\xi) = i_{\rho(\xi)}\Omega_a$$

$N = 2$ supersymmetric vacuum

Gaugino and gravitino variation (hyperino constrains vector mult.) [*Hristov, Looyestijin & Vandoren; Louis, Smyth & Triendl*]

$$\delta\psi_\mu = \delta\lambda = 0 \qquad \Longleftrightarrow \qquad \mu_a(\xi) = 0 \quad \forall\xi$$

form hyper-Kähler quotient

$$X' = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/G$$

X' is still hyper-Kähler (still Swann bundle)

Can be infinite dimensional... (eg Atiyah–Bott)

- $X =$ space of gauge connections on Riemann surface
- $G =$ group of gauge transformations
- $\mu = F$, field strength
- $X' =$ moduli space of flat connections

Hitchin's equations, Hermitian-Yang-Mills equations, Kähler-Einstein (Donaldson), ...

$E_{7(7)}\times\mathbb{R}^+$ generalised geometry [*CSW, from Hitchin, Gualtieri*]

Type II warped dimensional reduction

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{3,1}) + \mathrm{d}s^2_6(M),$$

all fields $\{g,\phi,B,\tilde{B},C^\pm,\Delta\}$ on M

Goal: new geometrical description

- unify bosonic symmetries and fields into single objects
- enhanced local $SU(8)$ symmetry

[*c.f. Julia, de Wit & Nicolai, . . . , Siegel; Hull, Hohm & Zweibach, . . . , Berman & Perry, . . .*]

Generalised tangent space [*Hull; Pacheco & DW*]

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \Lambda^\pm T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)$$
$$V^M = (v^m, \lambda_m, \tilde{\lambda}_{m_1\cdots 5}, \lambda^\pm_{\dots}, \dots)$$

parametrises infinitesimal symmetries – diffeos and gauge transf

$E_{7(7)}\times\mathbb{R}^+$ structure

Unique $E_{7(7)}\times\mathbb{R}^+ \supset GL(6,\mathbb{R})^\pm$ action on frames $V^M = V^A \hat{E}_A^M$

$$E \sim \mathbf{56_1}$$

where $\mathbb{R}^+$ weight is $\mathbf{1_p} \sim (\det T^*M)^{p/2}$

Generalised tensors: $E_{7(7)} \times \mathbb{R}^+$ representations

For example, adjoint includes potentials

$$\mathbf{133}_0 \sim (TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^\pm T^*M \oplus \dots$$
$$A^M{}_N = (a^m{}_n, B_{mn}, \tilde{B}_{m_1 \dots m_6}, C^\pm_{\dots}, \dots)$$

Dorfman derivative

Given $V \in E$, there is a generalisation of Lie derivative

$$L_V = \text{diffeo.} + \text{gauge transformation}$$

Generalised geometry and supergravity [CSW]

Generalised metric

$$G_{MN} \qquad \text{invariant under } SU(8) \subset E_{7(7)} \times \mathbb{R}^+$$

equivalent to $\{g, \phi, B, \tilde{B}, C^{\pm}, \Delta\}$

Generalised connection (c.f. Levi-Civita)

Define $D_M V^N = \partial_M V^N + \Omega_M{}^N{}_P V^P$

exists gen. torsion-free connection D with $DG = 0$

but not unique, (torsion $\sim \mathbf{912} + \mathbf{56}$)

Bosonic action

Analogue of Ricci tensor is unique

$$S_B = \int_M |\mathrm{vol}_G|\, R \qquad \text{eom} = \text{gen. Ricci flat}$$

where $|\mathrm{vol}_G| = (\det G)^{-1/28} = \sqrt{g}\mathrm{e}^{2\Delta}$

Leading-order fermions and supersymmetry

$$\not{D}\psi + D \curlyvee \rho = 0 \qquad \delta\psi = D \curlyvee \epsilon \qquad \text{etc}$$

unique operators, full theory has local $SU(8)$ invariance

Hypermultiplet structures

Conventional G structures

$$g \in GL(d,\mathbb{R})/O(d)$$
$$J \in GL(2n,\mathbb{R})/Sp(n)$$
$$\Omega \in GL(2n,\mathbb{R})/SL(n,\mathbb{C})$$

$E_{7(7)} \times \mathbb{R}^+$ generalised structures

$$G \in \mathbb{R}^+ \times E_{7(7)} / SU(8)$$

Generalised complex structures [*Hitchin, Gualtieri*]

Take $E = TM \oplus T^*M$

$$\Phi^\pm \in \mathbb{R}^+ \times O(6,6)/SU(3,3)$$

spinor $\Phi^\pm \in S^\pm(E) \simeq \Lambda^\pm T^*M$

$$\begin{aligned} J &\longrightarrow \Phi^+ = \mathrm{e}^{-\phi}\mathrm{e}^{-B-\mathrm{i}J} \\ \Omega &\longrightarrow \Phi^- = \mathrm{e}^{-\phi}\mathrm{e}^{-B}(\Omega_1 + \Omega_3 + \Omega_5) \end{aligned}$$

if no RR fields $N=2$ implies integrability $\mathrm{d}\Phi^\pm = 0$ [*GMPT*]

Hypermultiplet structures [*GLSW*]

$$\{J_a\} \in \mathbb{R}^+ \times E_{7(7)} \,/\, Spin^*(12)$$

where $\{J_a\} \sim \mathbf{133_1}$ are $SU(2)_R$ triplet

$$[J_a, J_b] = 2\kappa\epsilon_{abc}J_c$$
$$\mathrm{Tr}\,(J_a J_b) = -\kappa^2\delta_{ab} \in \det T^*M$$

for Calabi–Yau gives Ω in type IIA and J in type IIB

Example: complex structure in IIA

$$\mathbf{133}_0 \sim (TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^- T^*M \oplus \dots$$
$$A^M{}_N = (a^m{}_n, B_{mn}, \tilde{B}_{m_1 \dots m_6}, C_m, C_{mnp}, \dots)$$

defines triplet with $J_\pm = J_1 \pm \mathrm{i} J_2$

$$J_+ \sim \Omega \qquad J_- \sim \bar{\Omega} \qquad J_3 \sim I$$

where $I^2 = -1$.

Space of hypermultiplet structures X

Infinite-dimensional space of sections

$$\{J_a(x)\} \in X$$

tangent space $\{v_a(x)\} \in TX$ where

$$v_a(x) = \delta J_a(x) = [\alpha(x), J_a(x)] \qquad \alpha(x) \in \mathfrak{e}_{7(7)} + \mathbb{R}$$

hyper-Kähler (Swann space) structure (c.f. Wolf space)

$$\Omega_a(v, w) = \epsilon_{abc} \int_M \mathrm{Tr}\,(v_b w_c)$$

Relation to gauged $N = 2$ supergravity

Rewrite type II supergravity as $N = 2$, $d = 4$ theory

- keep all KK modes
- X is infinite dimensional space of hypermultiplets

for supersymmetric vacuum we need to know

what is the gauging?

Momentum maps

$$G = \text{diffeos and gauge transformations}$$

Infinitesimally generated by the Dorfman derivative, so $V \in E$ parameterise Lie algebra

$$\rho(V) = \{L_V J_a\} \in TX$$

and we find

$$\mu_a(V) = -\tfrac{1}{2}\epsilon_{abc} \int \mathrm{Tr}\,(J_b\, L_V J_c)$$

Example: complex structure in IIA

With $V = (v, \lambda, \tilde{\lambda}, \lambda^+, \tau)$

$$\mu_+(V) \sim \int \Omega \wedge \mathrm{d}\lambda_2$$

$$\mu_3(V) \sim \int \mathcal{L}_v \Omega \wedge \bar{\Omega} - \Omega \wedge \mathcal{L}_v \bar{\Omega}$$

Integrability and moduli space

$N = 2$ supersymmetric vacuum integrability conditions just

$$\mu_a(V) = 0 \qquad \text{for all } V \in E$$

Since structures related by a diffeo or gauge transformation are equivalent

$$\mathcal{M} = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / G$$

gives the moduli space of structures, and is automatically hyper-Kähler (Swann) space

Examples

$O(6,6)$ decomposition

It is useful to use the $SL(2,\mathbb{R}) \times O(6,6) \subset E_{7(7)}$ decomposition

$$\mathbf{133} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}^{\pm})$$
$$\alpha = (\alpha^i{}_j, \alpha^A{}_B, \alpha^{i\pm})$$

where

$$\mathbf{2} \sim (\det T^*M)^{-1/2} \oplus (\det T^*M)^{1/2}$$
$$\mathbf{12} \sim TM \oplus T^*M$$
$$\mathbf{32}^{\pm} \sim \Lambda^{\pm} T^*M$$

Pure spinor

$$J_+ = J_1 + \mathrm{i}J_2 = (0, 0, u^i \Phi^\pm) \qquad u^i = \begin{pmatrix} \mathrm{i}\kappa \\ -\kappa^{-1} \end{pmatrix}$$

defines $\{J_a\}$ where $\kappa^2 = \mathrm{i}\langle \Phi^\pm, \bar{\Phi}^\pm \rangle$ (Mukai pairing)

$$\mu_+(V) = \int \langle \Lambda^\mp, \mathrm{d}\Phi^\pm \rangle$$

$$\mu_3(V) = \int \langle \mathrm{d}\bar{\Phi}^\pm, i_v \Phi^\pm \rangle - \langle \bar{\Phi}^\pm, i_v \mathrm{d}\Phi^\pm \rangle - \langle \mathrm{d}\bar{\Phi}^\pm, \Lambda \wedge \Phi^\pm \rangle - \langle \bar{\Phi}^\pm, \Lambda \wedge \mathrm{d}\Phi^\pm \rangle$$

vanishes iff $\mathrm{d}\Phi^\pm = 0$

D3 brane on $HK_4 \times \mathbb{R}^2$

$$\mathrm{d}s^2(M) = \mathrm{e}^{2A}\left(\mathrm{d}s^2_{\mathrm{HK}} + \mathrm{d}x^2 + \mathrm{d}y^2\right) \qquad \Delta \neq 0 \qquad C_4 \neq 0$$

then $\{J_a\} = \{\mathrm{e}^{C_4}\widehat{J}_a\}$ where ${\widehat{J}_a}{}^i{}_j = 0$ and

$${\widehat{J}_a}{}^A{}_B = \tfrac{1}{2}\kappa\begin{pmatrix} j_a & 0 \\ 0 & -j_a^T \end{pmatrix} \qquad {\widehat{J}_a}{}^{i+} = \tfrac{1}{2}\kappa\begin{pmatrix} \mathrm{e}^{4A}\omega_a \wedge \mathrm{d}x \wedge \mathrm{d}y \\ -\mathrm{e}^{-4A}\kappa^{-2}\omega_a \end{pmatrix}$$

and

$$\mu(V)_a = 0 \qquad \text{iff} \qquad A = \Delta, \quad F_5 = \tfrac{1}{4} * \mathrm{d}(\mathrm{e}^{-4\Delta})$$

Vector multiplet

$$K \in \mathbb{R}^+ \times E_{7(7)} / E_{6(2)}$$

where $K \sim \mathbf{56_1}$ with compatibility

$$J_a \cdot K = 0, \qquad \sqrt{Q(K)} = \kappa^2$$

Infinite-dimensional special Kähler metric with integrability

$$L_{K+\mathrm{i}\hat{K}} J_a = 0$$

"K is generalised tri-holomorphic Killing vector"

Conclusions

- same construction works for $N=2$ reductions of 11d supergravity, just different decomposition

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$

- series of HK hypermultiplet structure spaces

$$\mathbb{R}^+ \times E_{7(7)} \,/\, Spin^*(12)$$
$$\mathbb{R}^+ \times E_{6(6)} \,/\, SU^*(6)$$
$$\mathbb{R}^+ \times Spin(5,5)/\, SU(2) \times Spin(1,5)$$

for reductions to 4, 5 and 6 dim from IIA/B or 11d

- conventionally

$$d\Omega = 0 \quad + \quad \text{mod out diffeos}$$

  here HK-quotient encodes both

- first example of

  full diffeo group in quotient

  (gauge transformation, Hamiltonian symplectomorphism, ...)

- extending to generalised structures simplifies moduli space problem (no obstructions, ...)

- for infinitesimal deformations, part of complex

$$\dots \longrightarrow E \xrightarrow{L.J_a} TX \xrightarrow{\mathrm{d}\mu_a} E^* \otimes \mathfrak{su}(2) \longrightarrow \dots.$$

elliptic ($\mathcal{M}$ finite dimensional)? cohomology?

- U-duality extension A and B top string

Kähler/Kodaira-Spencer gravity $\longrightarrow$ moment map

- generic 5d $N = 1$ AdS/CFT backgrounds ...