N=2 String Amplitudes & the Ω-background

Ioannis G. Florakis

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Based on work with

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Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

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MAX-PLANCK-GESELLSCHAFT

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- Insight into non-perturbative physics

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- Theories with a metric, but "physical" correlation functions are metric independent : topological !
- Existence of a fermionic symmetry operator Q : nilpotent

$$Q^2 = 0$$

- "Physical" operators are defined to be Q-closed

$$\{Q, \mathcal{O}_{\rm phys}\} = 0$$

- They lie in the cohomology of Q
- The energy-momentum tensor of the theory is Q-exact

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Correlation functions of physical operators are metric independent !

Topological String Theory

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- Start with N=(2,2) worldsheet SCFT $\{T, G^+, G^-, J\}$
- Twist by redefining the energy momentum tensor to eliminate the central charge

$$T(z) \to T(z) - \frac{1}{2}\partial J(z)$$

- All operators have their conformal weights shifted by -q/2
- Identify the superpartner of T with +1 U(1) charge and h=1with topological BRST current

$$Q = \oint G^+$$

- The energy-momentum tensor of the twisted theory becomes BRST-exact

$$\{Q, G^-(z)\} = \oint G^+ \cdot G^-(z) = T(z)$$

- Identify G^- with the reparametrization ghost b
- For a Riemann surface of genus g, there are 3(g-1) Beltrami differentials which, together with $G^-(z)$ form the invariant integration measure over moduli space

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3(g-1)} |G^-(\mu_i)|^2 \rangle$$

Topological String partition function

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J current is anomalous

$$T(z)J(w) = \frac{\hat{c}/2}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}$$

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introduces $\hat{c}(g-1)/2$ units of U(1) background charge, providing a possible cancellation of the U(1) charge of the measure !

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The topological partition function is non-vanishing for any genus for $\hat{c}/2 = 3$ (CY$_3$)

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graviphoton vertex operator in (-1/2,-1/2) ghost picture

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- 3(g-1) G- insertions already present

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Antoniadis, Gava, Narain, Taylor 1993

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$$\int d^4x \; d^4\theta \; F_g(X) \, (W_{\mu\nu}^{ij} W_{ij}^{\mu\nu})^g = \int d^4x \; F_g(\phi) R_{(-)\mu\nu\rho\sigma} R_{(-)}^{\mu\nu\rho\sigma} \; (F_{(-)\lambda\tau}^{G} F_{(-)}^{G,\lambda\sigma})^{g-1} + \ldots$$

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4d N=2 supergravity multiplet : Weyl chiral superfield

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the coupling function $F_g(X)$ depends only on holomorphic vector multiplets

$$X^I = \phi^I + \theta^i \lambda^I_i + \tfrac{1}{2} F^I_{(-)\mu\nu} \epsilon_{ij} (\theta^i \sigma^{\mu\nu} \theta^j) + \dots$$

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- Example: 4d N=2 Heterotic String Theory on K3xT^2
- Melvin-like fibration $ds^2 = (dX^\mu + \Omega^\mu dX + \bar{\Omega}^\mu d\bar{X})^2 + dX d\bar{X}$
- X : complexified T^2 coordinate
- X^μ : 4d spacetime coordinates
- Omega background : $d\Omega = \epsilon_1\, dX^1 \wedge dX^2 + \epsilon_2\, dX^3 \wedge dX^4$
- 2 parameters : $\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$

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No convincing proposal for refinement at the worldsheet level so far (twisted CFT)...

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2 proposals in the literature...

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- Go to dual theory : N=2 Heterotic string on K3xT^2
- Consider scattering amplitudes of the form: $F_{g,n} = \langle\, R^2_{(-)} (F^G_{(-)})^{2g-2} V^{2n}_{(+)} \,\rangle$
- First appear at genus 1 in the Heterotic theory (+corrections)
- Weak coupling limit : captures perturbative part of the genus-g Type II amplitude
- For n=0, one recovers the Topological String partition function $F_{g,0}=F_g$
- (-) : anti-self-dual fields whereas (+) : self-dual
- couple to either $SU_-(2)$ or $SU_+(2)$ factors of the 4d Lorentz group

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- Field strengths of vector partners of $\bar{T}, \bar{U}$ moduli + FI terms
- Claims to reproduce the Nekrasov partition function
- Cannot be evaluated exactly as a string amplitude
- Higher ϵ-corrections
- Non-compact limit only

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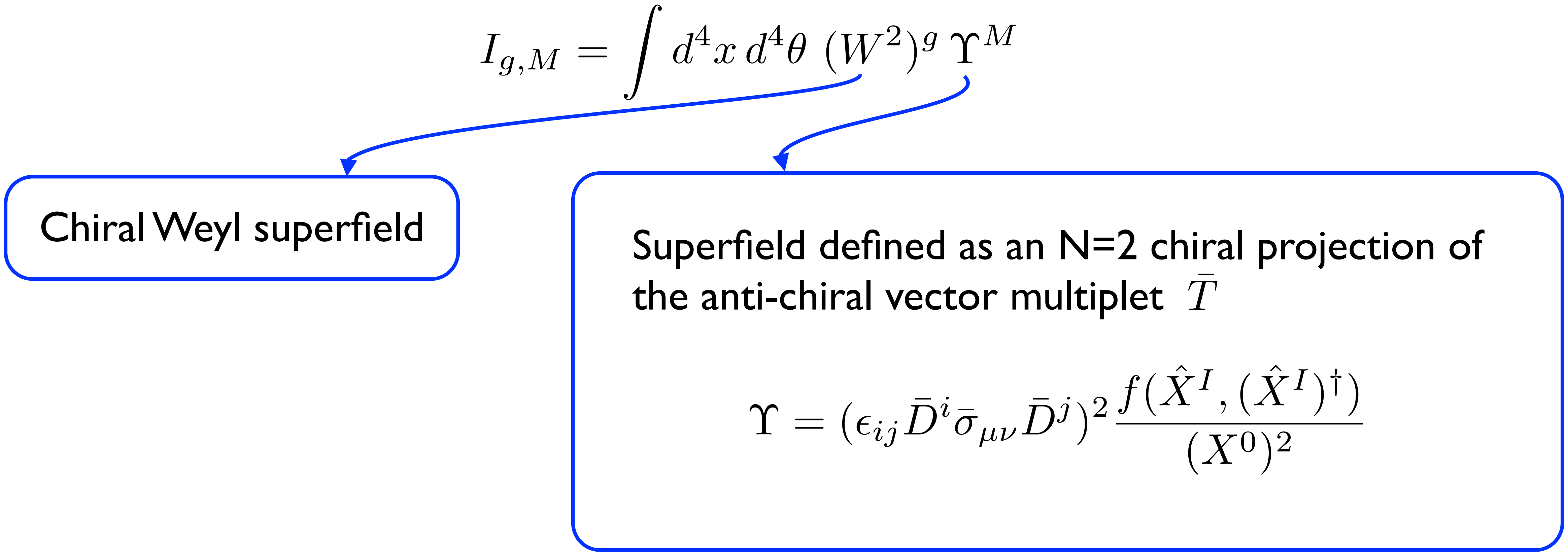

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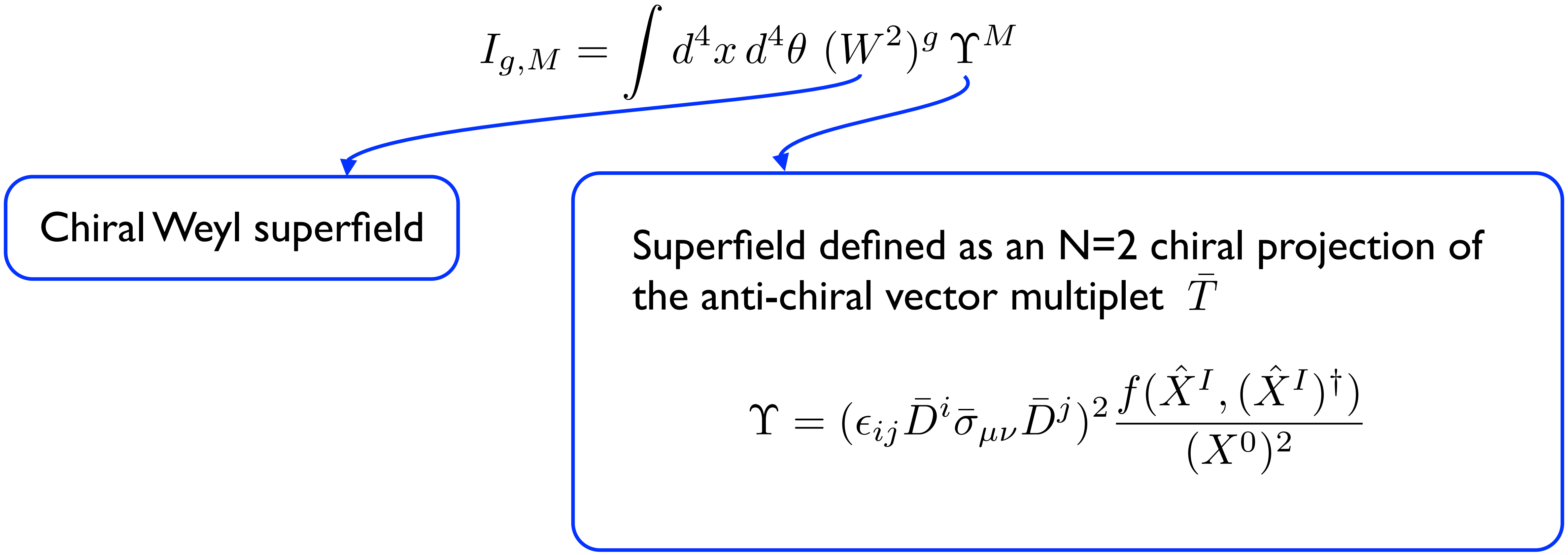

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- Technically more convenient to replace 2 gravitons + 2 graviphotons by 4 gravitini

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$$\langle R^2_{(-)}\,(F^G_{(-)})^{2g-2}\,(F^{\bar{T}}_{(+)})^{2M}\rangle \longrightarrow \langle B^4_{(-)}\,(F^G_{(-)})^{2g-2}\,(F^{\bar{T}}_{(+)})^{2M}\rangle$$

- We only capture the perturbative part of the refined amplitudes
- 4d N=2 Heterotic string on K3xT^2 in the presence of Wilson lines
- Technically more convenient to replace 2 gravitons + 2 graviphotons by 4 gravitini
- Z^1 , Z^2 , χ^1 , χ^2 : complexified spacetime

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$$V_{\psi^\pm}(\xi_{\mu,\alpha},p) = \xi_{\mu,\alpha}\, e^{-\varphi/2}S^\alpha e^{i\phi_3/2}\Sigma^\pm\bar{\partial}Z^\mu e^{ip\cdot Z}$$

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$$S^1 = e^{\frac{i}{2}(\phi_1+\phi_2)} \qquad S^2 = e^{-\frac{i}{2}(\phi_1+\phi_2)} \qquad \Sigma^\pm = e^{\pm\frac{i}{2}(\phi_4+\phi_5)}$$

Vertex Contributions

$$\left\langle (V_{\psi^+}(x_1) \cdot V_{\psi^+}(x_2)) \, (V_{\psi^-}(y_1) \cdot V_{\psi^-}(y_2)) \, (V^G(\epsilon_1, p_2) V^G(\epsilon_{\bar{1}}, p_{\bar{2}}))^N (V^{\bar{T}}(\epsilon_1, p_{\bar{2}}) V^{\bar{T}}(\epsilon_{\bar{1}}, p_2))^M \right\rangle$$

Vertex Contributions

Pick a particular kinematic configuration :

$$\left\langle (V_{\psi^+}(x_1)\cdot V_{\psi^+}(x_2))\,(V_{\psi^-}(y_1)\cdot V_{\psi^-}(y_2))\,(V^G(\epsilon_1,p_2)V^G(\epsilon_{\bar{1}},p_{\bar{2}}))^N(V^{\bar{T}}(\epsilon_1,p_{\bar{2}})V^{\bar{T}}(\epsilon_{\bar{1}},p_2))^M\right\rangle$$

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Let $2m \leq 2M$ of the F^T vertex operators contribute the fermion bilinear piece

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Pick a particular kinematic configuration :

$$\left\langle \left(V_{\psi^+}(x_1)\cdot V_{\psi^+}(x_2)\right)\left(V_{\psi^-}(y_1)\cdot V_{\psi^-}(y_2)\right)\left(V^G(\epsilon_1,p_2)V^G(\epsilon_{\bar{1}},p_{\bar{2}})\right)^N\left(V^{\bar{T}}(\epsilon_1,p_{\bar{2}})V^{\bar{T}}(\epsilon_{\bar{1}},p_2)\right)^M\right\rangle$$

Let $2m \leq 2M$ of the F$^{\mathrm{T}}$ vertex operators contribute the fermion bilinear piece

field	**pos.**	**number**	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	**bosonic**
gravitino	x_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$Z^1\bar{\partial}Z^2$
	x_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$\bar{Z}^1\bar{\partial}\bar{Z}^2$
	y_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$Z^1\bar{\partial}Z^2$
	y_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\bar{Z}^1\bar{\partial}\bar{Z}^2$
F^G	z	N	0	0	0	0	0	$\partial X\ Z^1\bar{\partial}Z^2$
	z'	N	0	0	0	0	0	$\partial X\ \bar{Z}^1\bar{\partial}\bar{Z}^2$
$F^{\bar{T}}$	u	m	$+1$	-1	0	0	0	$\bar{\partial}X$
	u'	m	-1	$+1$	0	0	0	$\bar{\partial}X$
	t	$M-m$	0	0	0	0	0	$\bar{\partial}X\ Z^1\partial\bar{Z}^2$
	t'	$M-m$	0	0	0	0	0	$\bar{\partial}X\ \bar{Z}^1\partial Z^2$
PCO	P	2	0	0	-1	0	0	∂X

Vertex Contributions

Pick a particular kinematic configuration :

$$\left\langle (V_{\psi^+}(x_1)\cdot V_{\psi^+}(x_2))\,(V_{\psi^-}(y_1)\cdot V_{\psi^-}(y_2))\,(V^G(\epsilon_1,p_2)V^G(\epsilon_{\bar{1}},p_{\bar{2}}))^N(V^{\bar{T}}(\epsilon_1,p_{\bar{2}})V^{\bar{T}}(\epsilon_{\bar{1}},p_2))^M\right\rangle$$

Let $2m \leq 2M$ of the F^T vertex operators contribute the fermion bilinear piece

field	pos.	number	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	bosonic
gravitino	x_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$Z^1\bar{\partial}Z^2$
	x_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$\bar{Z}^1\bar{\partial}\bar{Z}^2$
	y_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$Z^1\bar{\partial}Z^2$
	y_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\bar{Z}^1\bar{\partial}\bar{Z}^2$
F^G	z	N	0	0	0	0	0	$\partial X\, Z^1\bar{\partial}Z^2$
	z'	N	0	0	0	0	0	$\partial X\, \bar{Z}^1\bar{\partial}\bar{Z}^2$
$F^{\bar{T}}$	u	m	$+1$	-1	0	0	0	$\bar{\partial}X$
	u'	m	-1	$+1$	0	0	0	$\bar{\partial}X$
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	t'	$M-m$	0	0	0	0	0	$\bar{\partial}X\, \bar{Z}^1\partial Z^2$
PCO	P	2	0	0	-1	0	0	∂X

Include 2 PCOs to cancel the -2 units of superghost charge of the gravitini

Vertex Contributions

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field	pos.	number	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	bosonic
gravitino	x_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$Z^1\bar\partial Z^2$
	x_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$\bar Z^1\bar\partial \bar Z^2$
	y_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$Z^1\bar\partial Z^2$
	y_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\bar Z^1\bar\partial \bar Z^2$
F^G	z	N	0	0	0	0	0	$\partial X\ Z^1\bar\partial Z^2$
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	t	$M-m$	0	0	0	0	0	$\bar\partial X\ Z^1\partial \bar Z^2$
	t'	$M-m$	0	0	0	0	0	$\bar\partial X\ \bar Z^1\partial Z^2$
PCO	P	2	0	0	-1	0	0	∂X

Include 2 PCOs to cancel the -2 units of superghost charge of the gravitini

The 2 PCOs contribute with their internal part only, in order to cancel the ϕ_3 U(1) charge

Bosonic Generating Functions

Bosonic Generating Functions

The bosonic part of the correlator is

$$\langle (Z^1\bar{\partial}Z^2)^{N+2}(\bar{Z}^1\bar{\partial}\bar{Z}^2)^{N+2}(Z^1\partial\bar{Z}^2)^{M-m}(\bar{Z}^1\partial Z^2)^{M-m}(\partial X)^{2N+2}(\bar{\partial}X)^{2M}\rangle$$

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It can be generated by taking ε-derivatives of the generating function :

$$G^{\mathrm{bos}}(\epsilon_-,\epsilon_+) = \Big\langle \exp\Big[-\epsilon_-\int d^2z\;\partial X(Z^1\bar{\partial}Z^2+\bar{Z}^2\bar{\partial}\bar{Z}^1)-\epsilon_+\int d^2z\;(Z^1\partial\bar{Z}^2+Z^2\partial\bar{Z}^1)\bar{\partial}X\Big]\Big\rangle$$

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- ε- derivatives of G_{bos}(ε-,ε+) produce powers of N

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- ε_- derivatives of $G_{\rm bos}(\varepsilon_-,\varepsilon_+)$ produce powers of N
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- ε_- derivatives of $G_{bos}(\varepsilon_-,\varepsilon_+)$ produce powers of N
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Since only X appears in the correlator but not its complex conjugate, the T^2 currents only contribute zero modes

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- ε+ derivatives of $G_{bos}(ε_-,ε_+)$ produce powers of M

Since only X appears in the correlator but not its complex conjugate, the T^2 currents only contribute zero modes

$$G^{\rm bos}(\epsilon_-,\epsilon_+) = \Big\langle \exp\Big[-\tilde\epsilon_-\int d^2z\ (Z^1\bar\partial Z^2+\bar Z^2\bar\partial\bar Z^1)-\check\epsilon_+\int d^2z\ (Z^1\partial\bar Z^2+Z^2\partial\bar Z^1)\Big]\Big\rangle$$

where we absorb the dX zero modes into the deformation parameters ε via:

$$\begin{cases}\tilde\epsilon_\pm = \langle\partial X\rangle\epsilon_\pm = \lambda_i(M+\bar\tau N)^i\epsilon_\pm \\ \check\epsilon_\pm = \langle\bar\partial X\rangle\epsilon_\pm = \bar\lambda_i(M+\tau N)^i\epsilon_\pm\end{cases}$$

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- ε- derivatives of $G_{\rm bos}$(ε-,ε$_+$) produce powers of N
- ε$_+$ derivatives of $G_{\rm bos}$(ε-,ε$_+$) produce powers of M

Since only X appears in the correlator but not its complex conjugate, the T^2 currents only contribute zero modes

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the path integral is effectively Gaussian !

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Hence, under modular transformations $\tau \to -1/\tau$
the redefined parameters transform covariantly

Bosonic Generating Functions

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- ε- derivatives of $G_{\mathrm{bos}}(\varepsilon_-,\varepsilon_+)$ produce powers of N
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Since only X appears in the correlator but not its complex conjugate, the T^2 currents only contribute zero modes

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Hence, under modular transformations $\tau\to -1/\tau$
the redefined parameters transform covariantly

$$\boxed{\begin{aligned} \tilde{\epsilon}_\pm &\to \frac{\tilde{\epsilon}_\pm}{\bar{\tau}} \\ \check{\epsilon}_\pm &\to \frac{\check{\epsilon}_\pm}{\tau} \end{aligned}}$$

Bosonic Generating Functions

Bosonic Generating Functions

After Poisson resummation to go to the Hamiltonian representation

$$\begin{cases} \tilde{\epsilon}_{\pm} = [(T-\bar{T})(U-\bar{U}) - \frac{1}{2}(Y-\bar{Y})^2]^{-1/2} \ \tau_2 P_L \epsilon_{\pm} \\ \check{\epsilon}_{\pm} = [(T-\bar{T})(U-\bar{U}) - \frac{1}{2}(Y-\bar{Y})^2]^{-1/2} \ \tau_2 P_R \epsilon_{\pm} \end{cases}$$

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Expand the bosonic fields into modes and perform the path integral

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Expand the bosonic fields into modes and perform the path integral

$$G^{\rm bos}(\epsilon_-,\epsilon_+) = G_{\rm hol}(\epsilon_-,\epsilon_+) \times G_{\text{non-hol}}(\epsilon_-,\epsilon_+)$$

Bosonic Generating Functions

After Poisson resummation to go to the Hamiltonian representation

$$\begin{cases} \tilde{\epsilon}_\pm = [(T-\bar{T})(U-\bar{U}) - \tfrac{1}{2}(Y-\bar{Y})^2]^{-1/2}\ \tau_2 P_L \epsilon_\pm \\ \check{\epsilon}_\pm = [(T-\bar{T})(U-\bar{U}) - \tfrac{1}{2}(Y-\bar{Y})^2]^{-1/2}\ \tau_2 P_R \epsilon_\pm \end{cases}$$

Expand the bosonic fields into modes and perform the path integral

$$G^{\rm bos}(\epsilon_-,\epsilon_+) = G_{\rm hol}(\epsilon_-,\epsilon_+) \times G_{\text{non-hol}}(\epsilon_-,\epsilon_+)$$

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- for $\beta=2\alpha>2$, it converges to the non-holomorphic Poincaré series of weight 0

$$\lim_{\kappa\to 0} \Phi_{\alpha,2\alpha} = 2\zeta(2\alpha)E(\alpha;\tau)$$

- for $\beta=2\alpha=2$, $\Phi_{1,2}$ is singular and admits no analytic continuation to $\kappa=0$
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Fourier expansion

$$\tau_2^{-\alpha}\Phi_{\alpha\beta}(0;\tau,\bar\tau) = 2\zeta(\beta) + 2\tau_2^{1-\beta}\left\{ C_0^{\alpha,\beta} + \sum_{n>0}\left[C_n^{\alpha,\beta}(\tau_2)q^n + I_n^{\alpha,\beta}(\tau_2)\bar{q}^n \right] \right\}$$

$$C_n^{\alpha,\beta}(\tau_2) = \frac{(2\pi)^\beta(-i)^{\beta-2\alpha}}{\Gamma(\beta-\alpha)}(n\tau_2)^{\beta-1}\sigma_{1-\beta}(n)(4\pi n\tau_2)^{-\frac{\beta}{2}}e^{2\pi n\tau_2}W_{\frac{\beta}{2}-\alpha,\frac{\beta-1}{2}}(4\pi n\tau_2)$$

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- Here, fermions are charged and can give non-trivial contribution in the field theory limit

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non-trivial K3-fermion correlator

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$$G(\epsilon_-, \epsilon_+) = G^{\mathrm{bos}}(\epsilon_-, \epsilon_+)\, \frac{1}{\eta^4 \bar{\eta}^{24}}\, \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} G^{\mathrm{ferm}} \begin{bmatrix} h \\ g \end{bmatrix} (\check{\epsilon}_+)\, Z \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{(2,2+8)}(T, U; Y)$$

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$$\Gamma_{(2,2+8)} = \sum_{m_i,n^i,Q^a\in\mathbb{Z}} q^{|P_L|^2}\,\bar q^{|P_R|^2+\frac{1}{2}\sum(Q^a-Y_i^a n^i)^2}$$

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$$\Gamma_{(2,2+8)} = \sum_{m_i,n^i,Q^a\in\mathbb{Z}} q^{|P_L|^2}\bar q^{|P_R|^2+\frac{1}{2}\sum(Q^a-Y^a_i n^i)^2}$$

- modular invariant

Full Amplitude

Putting all the pieces together, the full amplitude is

$$G(\epsilon_-,\epsilon_+) = G^{\rm bos}(\epsilon_-,\epsilon_+)\,\frac{1}{\eta^4\bar\eta^{24}}\,\frac{1}{N}\sum_{a,b\in\mathbb{Z}_N} G^{\rm ferm}\begin{bmatrix} h \\ g \end{bmatrix}(\check\epsilon_+)\,Z\begin{bmatrix} h \\ g \end{bmatrix}\Gamma_{(2,2+8)}(T,U;Y)$$

Z is the orbifold block of the K3 lattice together with the partition function of E$_7$xSU(2)

$\Gamma_{(2,2+8)}$ is the partition function of the T^2 and E$_8$ lattices deformed by the Wilson line

$$Z\begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{\rm K3}\begin{bmatrix} h \\ g \end{bmatrix}\,\frac{1}{2}\sum_{k,\ell=0}^{1}\bar\theta^6\begin{bmatrix} k \\ \ell \end{bmatrix}\bar\theta\begin{bmatrix} k+h \\ \ell+g \end{bmatrix}\bar\theta\begin{bmatrix} k-h \\ \ell-g \end{bmatrix}$$

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- ● modular invariant
- ● exactly calculable order by order

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$$Y_1^{a\star} = Y_2^{a\star} = (\tfrac{1}{2}, \tfrac{1}{2}, y^3, \dots, y^8)$$

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$Y = Y_2 - U Y_1$ complexified Wilson line

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the Z_N projected fermionic contribution produces the Nekrasov phase in the field theory limit !

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Including all contributions, we extract the amplitude at the SU(2) enhancement point

$$\mathcal{F}(\epsilon_-,\epsilon_+) \sim (\epsilon_-^2-\epsilon_+^2)\ \int_0^\infty \frac{dt}{t}\ \frac{-2\cos(2\epsilon_+ t)}{\sin(\epsilon_- - \epsilon_+)t\ \sin(\epsilon_- + \epsilon_+)t}\ e^{-\mu t}$$

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- BPS mass parameter $\mu \sim \bar{Y} \cdot Q$
- Leading singularity power of $F_{(g,n)}$ correctly given as $\mu^{2-2g-2n}$
- Reproduces precisely the perturbative part of Nekrasov's partition function in 4d !
- Holomorphic dependence on the complexified Wilson line Y
- Even powers of ε_+, ε_- : coupling to self- / anti-self dual field strengths (Lorentz invariance)
- Asymmetry under exchange of ε_+, ε_- : phase / R-symmetry twist

Radius Deformations & Nekrasov-Okounkov formula

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Before expanding around the enhancement point, take the volume of T^2 to be much larger than the string scale

$$T_2 = \mathrm{Vol}(T^2) \gg 1$$

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$$\sum_{\substack{g_1,g_2\geq 0\\ g_1+g_2\in 2\mathbb{Z}}} \frac{B_{g_1}B_{g_2}}{g_1!g_2!}\epsilon_1^{g_1-1}\epsilon_2^{g_2-1}\left(\frac{i\pi}{U_2}\right)^{g_1+g_2-2} \sum_{m_i}{}' e^{2\pi i(Y_i\cdot Q)m^i}(m_1+Um_2)^{g_1+g_2-2}\frac{U_2}{|m_1+Um_2|^2}$$

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Nekrasov and Okounkov considered the partition function of a 5d theory with 8 supercharges, compactified on a circle S^1 of radius β and with Ω-twist in the 4 non-compact dimensions

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$$\gamma_{\epsilon_1,\epsilon_2}(x|\beta) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-\beta x}}{(e^{\beta n \epsilon_1} - 1)(e^{\beta n \epsilon_2} - 1)}$$

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with the identifications :

$$\beta = 2\pi R_1 \qquad , \qquad x = -i\frac{Y_1 \cdot Q}{R_1}$$

Dual Type I Theory

Antoniadis, Partouche, Taylor 1998

Heterotic-Type I duality in D=4 on K3xT^2 : weakly coupled regimes

$$T = B_{45} + iG^{1/2} \longrightarrow S' = B_{45} + iG^{1/4}V^{-1/2}e^{-\phi_4}$$

- Realize K3 ~ T^4/Z_2 orbifold
- 16 D9 and 16 D5 branes at orbifold fixed point : U(16) x U(16)

Bianchi, Sagnotti 1991

- Wilson Lines for the D9's : study enhancement points
- S, S' associated with gauge couplings of the D9 and D5 branes

Antoniadis, IF, Hohenegger, Narain, Zein Assi 2013

Anti-self-dual graviphoton

$$\left[(\partial X + ip\cdot\chi\Psi)(\bar\partial Z^\mu + ip\cdot\tilde\chi\tilde\chi^\mu) - p_\nu(\sigma^{\mu\nu})^{\alpha\beta}e^{-(\phi+\tilde\phi)/2}S^\alpha\tilde S_\beta e^{i(\phi_3+\tilde\phi_3)/2}\Sigma^+\tilde\Sigma^-\right]e^{ip\cdot Z} + [\mathrm{L}\leftrightarrow\mathrm{R}]$$

Self-dual S'-vector (b=+1) or S-vector (b=-1)

$$\left[(\partial X + ip\cdot\chi\Psi)(\bar\partial Z^\mu + ip\cdot\tilde\chi\tilde\chi^\mu) + bp_\nu(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}e^{-(\phi+\tilde\phi)/2}S^{\dot\alpha}\tilde S_{\dot\beta} e^{i(\phi_3+\tilde\phi_3)/2}\hat\Sigma^+\hat{\tilde\Sigma}^-\right]e^{ip\cdot Z} + [\mathrm{L}\leftrightarrow\mathrm{R}]$$

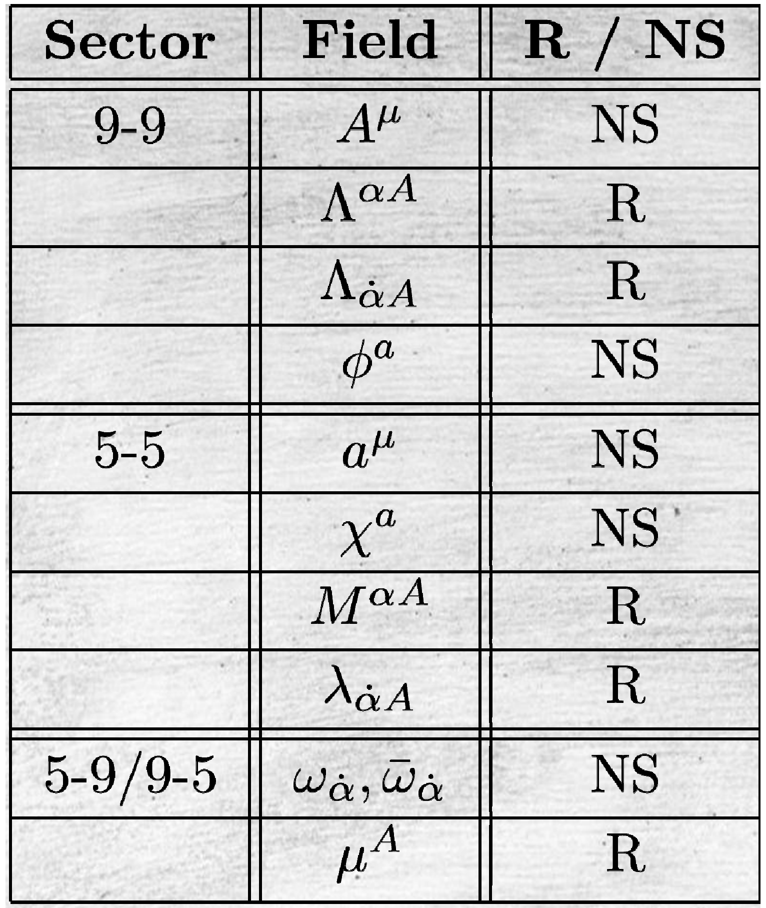
Dual Type I Theory

Antoniadis, IF, Hohenegger, Narain, Zein Assi 2013

One loop effective coupling, expanded around SU(2) enhancement point, correctly reproduces the perturbative part of Nekrasov's partition function

What about the non-perturbative part ?

- Realize gauge theory instantons as D-brane configurations (point-like in 4d)
- D9 gauge theory effective action contains $\mu_5 \int C_6 \wedge \mathrm{Tr}[\,F \wedge F\,]$
- Instanton configuration on D9 carries C_6 charge : D5 instanton
- N D9's and k D5-instantons wrapping entirely internal space
- 9-9 : perturbative N=2 SYM gauge theory U(N)
- 5-5 : unmixed instanton moduli
- 9-5 and 5-9 : mixed instanton moduli



Sector	Field	R / NS
9-9	A^μ	NS
	$\Lambda^{\alpha A}$	R
	$\Lambda_{\dot\alpha A}$	R
	ϕ^a	NS
5-5	a^μ	NS
	χ^a	NS
	$M^{\alpha A}$	R
	$\lambda_{\dot\alpha A}$	R
5-9/9-5	$\omega_{\dot\alpha}, \bar\omega_{\dot\alpha}$	NS
	μ^A	R

Instanton corrections

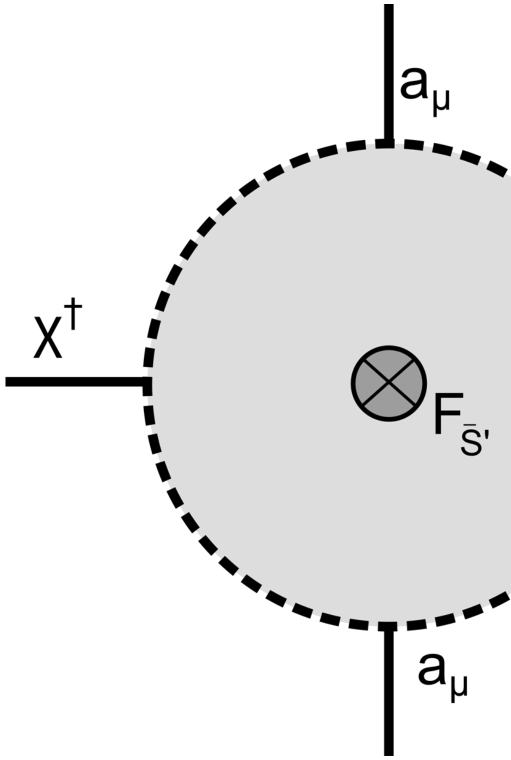
Study instanton configurations in the closed string background of graviphotons and S’-vectors

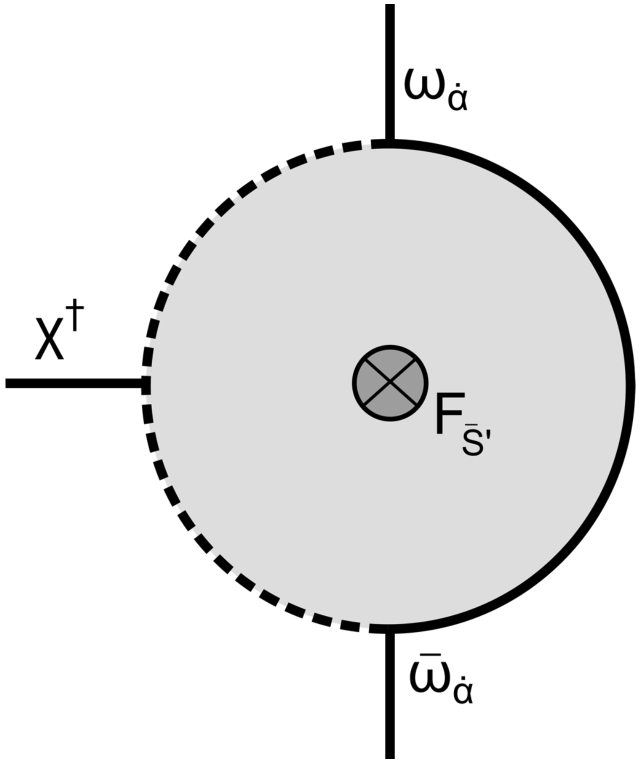
In pure graviphoton background ($\varepsilon_+=0$) reproduces Ω-deformed ADHM action

Billó, Frau, Fucito, Lerda 2006

Consider now full background with self-dual S’-vectors of field strength ~ ε_+

- Calculate all disc diagrams and take field theory limit
- D9-D9 diagrams : N=2 SYM action
- D5-D5 diagrams
- D9-D5 diagrams





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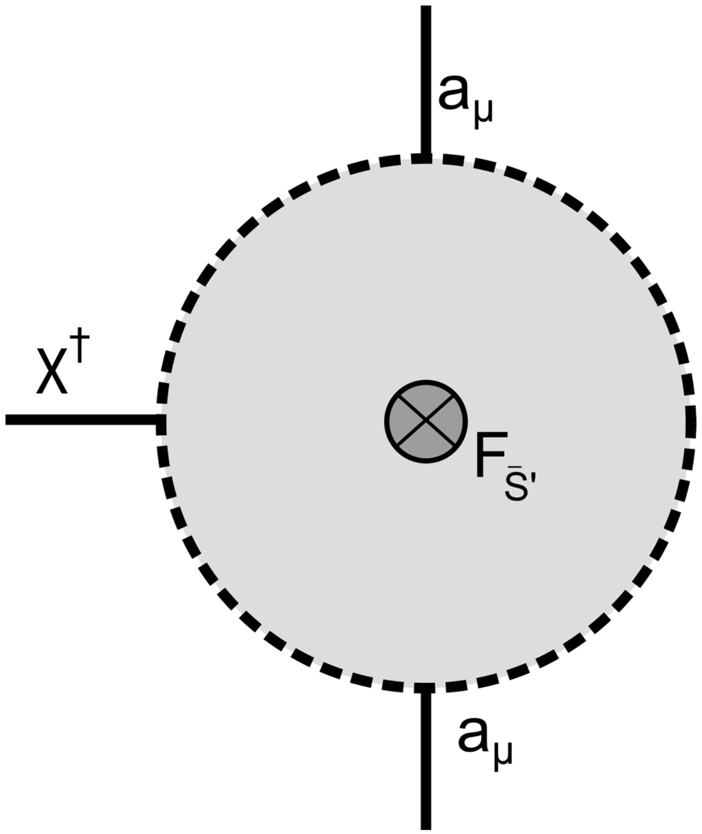
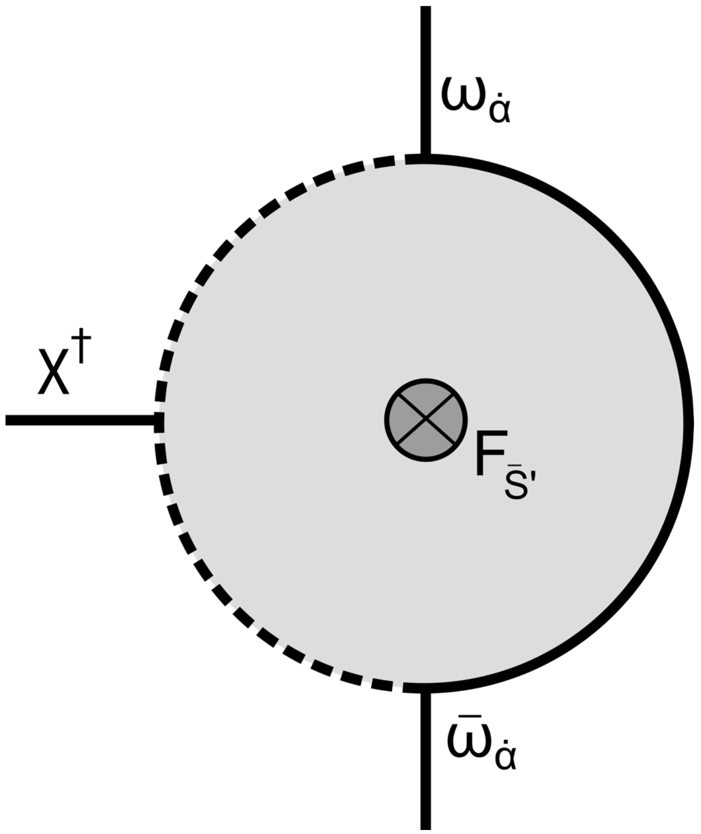


$$= -4i\,\mathrm{Tr}\Big([\chi^\dagger, a_\mu]\, a_\nu\, F^{\mu\nu}_{\bar{S}'}\Big)$$



$$= \frac{i}{2}\,\mathrm{Tr}\Big(\bar\omega_{\dot\alpha}\,\chi^\dagger\omega_{\dot\beta}(\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}F^{\bar{S}'}_{\mu\nu}\Big)$$

Instanton corrections

Antoniadis, IF, Hohenegger, Narain, Zein Assi 2013

Identify constant field strength of self-dual S'-vectors with ε₊

$$F^{\bar{S}'}_{\mu\nu} = \bar{\eta}^{c}_{\mu\nu}\,\delta_{3c}\,\frac{\epsilon_+}{2}$$

Adding all disc diagrams together reproduces the full Ω-deformed ADHM action

$$S_{\mathrm{ADHM}} = -\mathrm{Tr}\Big([\chi^\dagger, a_{\alpha\dot{\beta}}]([\chi, a^{\dot{\beta}\alpha}] + \epsilon_-(a\tau_3)^{\dot{\beta}\alpha}) - \chi^\dagger\bar{\omega}_{\dot{\alpha}}(\omega^{\dot{\alpha}}\chi - \tilde{a}\omega^{\dot{\alpha}}) - (\chi\bar{\omega}_{\dot{\alpha}} - \bar{\omega}_{\dot{\alpha}}\tilde{a})\omega^{\dot{\alpha}}\chi^\dagger$$
$$\underbrace{+\,\epsilon_+[\chi^\dagger, a_{\alpha\dot{\beta}}](\tau_3 a)^{\dot{\beta}\alpha}}_{\text{unmixed}}\;\underbrace{-\,\epsilon_+\bar{\omega}_{\dot{\alpha}}(\tau_3)^{\dot{\alpha}}{}_{\dot{\beta}}\chi^\dagger\omega^{\dot{\beta}}}_{\text{mixed}}\Big)$$

Integrating over instanton moduli yields the non-perturbative Nekrasov partition function

$$Z^{\mathrm{Nek}}(\epsilon_-, \epsilon_+) = Z_{\mathrm{pert}}(\epsilon_-, \epsilon_+)\, Z_{\text{non-pert}}(\epsilon_-, \epsilon_+)$$

Nekrasov 2002
Nekrasov, Okounkov 2003

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Outlook

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- ☑ Generalized Holomorphic anomaly equation ?
- ☑ Type II dual on K3-fibered CY threefold : twisted correlator ?

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Thank You !