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ALGEBRAIC ASPECTS OF NON-GEOMETRIC FLUX VACUA

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Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο

ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ ΚΑΙ ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ

OUTLINE

- **Toroidal flux vacua and their T-dual faces:**
  Closed string non-geometry $\leftrightarrow$ non-commutativity/non-associativity.

- **Magnetic field analogue of non-geometry in Maxwell-Dirac theory:**
  – **Non-commutativity/non-associativity of momenta induced by $\vec{B}$.**
  – **The fate of angular symmetry (classical and quantum).**

- **Deformation theory and cohomology:**
  3-cocycles in Lie algebra and group cohomology.

- **Construction of star-product in phase space:**
  – **Substitute for canonical quantization.**
  – **Application to T-dual faces of closed string flux vacua.**

- **Further generalizations, discussion and conclusions.**

TOROIDAL FLUX MODELS

It has been established that non-geometric closed string backgrounds exhibit non-commutativity/non-associativity among their coordinates.

The prime example is provided by backgrounds originating from torus T^3 **with constant** H**-flux**

$$H_3 = H \; dX^1 \wedge dX^2 \wedge dX^3$$

satisfying the standard quantization condition

$$\frac{1}{4\pi^2} \int H = k \,, \qquad k \in \mathbb{Z} \,.$$

Choose the components of the anti-symmetric B**-filed,** $H_3 = dB$**, as**

$$B_{12} = Hx^3, \qquad B_{23} = 0 = B_{31}$$

and perform T-duality, successively, thinking of T^3 as T^n fibration over the base space T^{3-n} for $n = 0, 1, 2, 3$.

• *H*-flux model, which is the original geometric background $(n = 0)$. The toroidal coordinates are commuting,

$$[x^i,\ x^j] = 0.$$

• *f*-flux model, which follows by performing T-duality in x^1-direction $T^1_{x^1}$ $(n = 1)$. The resulting background is the Heisenberg nilmanifold (twisted torus) and it is fully geometric. It exhibits the relations

$$[x^1,\ \tilde{x}^2] \sim f\tilde{p}^3.$$

• **Q-flux model, which follows by performing T-duality in x^1 and x^2 directions $T^2_{x^1,x^2}$ $(n=2)$. The resulting background is geometric only locally but not globally, since the fibre is glued by T-duality when transported around the base. It exhibits the commutation relations**

$$[x^1,\ x^2]\ \sim\ Q\tilde{p}^3.$$

– The geometry has already become non-commutative.

• **R-flux model, which follows by performing T-duality in all directions $T^3_{x^1,x^2,x^3}$ $(n=3)$. The resulting background is entirely non-geometric and it exhibits the commutation relations**

$$[x^1,\ x^2]\ \sim\ Rp^3.$$

– The geometry has become non-associative.

It is more convenient to work with symmetric choice of B-field

$$B_{12} = \frac{H}{3}x^3, \qquad B_{23} = \frac{H}{3}x^1, \qquad B_{31} = \frac{H}{3}x^2.$$

Then, the commutation relations among x^i and $\tilde{x}^i$ take the form

$$\mathbf{H}: \quad [x^i,\ x^j] = 0, \quad [x^i,\ \tilde{x}^j] = 0, \quad [\tilde{x}^i,\ \tilde{x}^j] = iH\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{f}: \quad [x^i,\ x^j] = 0, \quad [\tilde{x}^i,\ \tilde{x}^j] = 0, \quad [x^i,\ \tilde{x}^j] = if\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{Q}: \quad [\tilde{x}^i,\ \tilde{x}^j] = 0, \quad [x^i,\ \tilde{x}^j] = 0, \quad [x^i,\ x^j] = iQ\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{R}: \quad [\tilde{x}^i,\ \tilde{x}^j] = 0, \quad [x^i,\ \tilde{x}^j] = 0, \quad [x^i,\ x^j] = iR\epsilon^{ijk}p^k.$$

MAGNETIC FIELD ANALOGUE OF NC/NA

A spinless point-particle (e, m) in magnetic field background $\vec{B}(\vec{x})$ **has commutation relations among its coordinates and momenta:**

$$[x^i,\ p^j] = i\delta^{ij}, \qquad [x^i,\ x^j] = 0, \qquad [p^i,\ p^j] = ie\ \epsilon^{ijk} B_k(\vec{x})$$

leading to non-commutativity of p^i **in Maxwell theory,** $\vec{\nabla} \cdot \vec{B} = 0$**.**

In Dirac's generalization of Maxwell theory we have $\vec{\nabla} \cdot \vec{B} \neq 0$ **and**

$$[[p^i,\ p^j],\ p^k]\ +\ \text{cyclic}\ \equiv\ [p^i,\ p^j,\ p^k]\ = -e\ \epsilon^{ijk} \vec{\nabla} \cdot \vec{B}\ \neq\ 0$$

Associativity of momenta is lost in the presence of magnetic charges.

This provides a simple model for NC/NA of string theory with $x^i \leftrightarrow p^i$**.**

Consider a continuous spherically symmetric distribution of magnetic charge in space, $\rho(x)$, to study (some of) the implications of NC/NA in classical and quantum theory. Setting $x^2 = \vec{x} \cdot \vec{x}$, we have

$$\vec{\nabla} \cdot \vec{B} = \rho(x), \qquad \vec{\nabla} \times \vec{B} = 0 \quad \text{(static)}.$$

The particular solution of the inhomogeneous equation is expressed as

$$\vec{B}(\vec{x}) = \frac{\vec{x}}{f(x)}, \qquad \rho(x) = \frac{3f(x) - xf'(x)}{f^2(x)}.$$

Some notable example are:

- $f(x) = x^3/g$ **so that** $\rho(x) = 4\pi g\ \delta(x)$ **[Dirac monopole with charge g]**
- $f(x) = 3/\rho$ **so that** $\rho(x) = \rho$ **is constant [cf parabolic flux model]**

Study the dynamics of point-particle for general profile function f(x).

Using the Hamiltonian $H = \vec{p} \cdot \vec{p}/2m$, the Lorentz force acting on the spinless particle (e, m) in the magnetic field background is

$$\frac{d\vec{p}}{dt} \;=\; i[H,\; \vec{p}\,] \;=\; \frac{e}{2m}(\vec{p} \times \vec{B} - \vec{B} \times \vec{p}\;)$$

which for $\vec{B}(\vec{x}) = \vec{x}/f(x)$ takes the special form

$$\frac{d^2\vec{x}}{dt^2} = -\frac{e}{mf(x)}\left(\vec{x} \times \frac{d\vec{x}}{dt}\right).$$

Lorentz force is proportional to angular momentum and does no work.

Energy conservation provides one integral of motion, $E = A/2m$.
Simple manipulation shows that $x^2(t) = At^2 + D$, setting $x^2(0) = D$.
[D provides the closest distance to the origin (perihelion of trajectory)]
Complete integrability requires three more integrals of motion.

For general choices of profile function $f(x)$, however, there are no additional integrals, since

$$\frac{d}{dt}\left(\vec{x}\times\frac{d\vec{x}}{dt}\right) = -\frac{e}{mf(x)}\,\vec{x}\times\left(\vec{x}\times\frac{d\vec{x}}{dt}\right) = \frac{e\ x^3}{mf(x)}\,\frac{d\hat{x}}{dt}.$$

Angular symmetry is broken in the presence of magnetic charges!

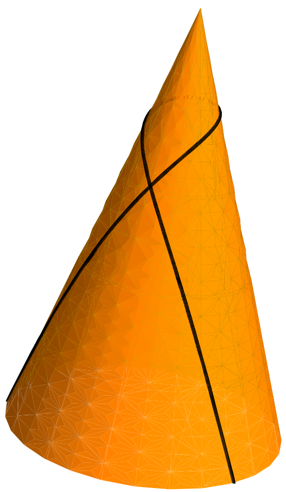
The only exception is the Dirac monopole having $f(x) = x^3/g$. In this case, the improved angular momentum $\vec{J} = m\vec{K}$ is conserved, where

$$\vec{K} \equiv \vec{x}\times\frac{d\vec{x}}{dt} - \frac{eg}{m}\,\hat{x}$$

is the celebrated Poincaré vector.

In all other case, including constant $f(x)$, angular symmetry is broken and the classical motion of the particle appears to be non-integrable.

Trajectory of a spinless particle in the field of a magnetic monopole: the charged particle (e, m) precesses with angular velocity $\vec{K}/(At^2 + D)$



- **The magnetic monopole g is located at the tip of the cone**
- **The Poincaré vector $\vec{K}$ provides the axis of the cone**
- $\vec{K} \cdot \hat{x} = -eg/m$ **determines the opening angle of the cone**

ANOTHER LOOK AT ANGULAR SYMMETRY

Try to follow as closely as possible the conventional definitions and algebraic structures of particle dynamics, without assuming particular representations nor Hilbert space [only that $\vec{p}$ acts as derivation].

Assume that $\vec{x}$ and $\vec{p}$ form a complete and irreducible set of observables for the point-particle in a static magnetic field $\vec{B}(\vec{x})$.

Angular momentum $\vec{J}$ ought to satisfy the algebraic relations

$$[J^i,\ x^j\ ] = i\epsilon^{ijk}\, x^k, \qquad [J^i,\ p^j\ ] = i\epsilon^{ijk}\, p^k, \qquad [J^i,\ J^j\ ] = i\epsilon^{ijk}\, J^k$$

so that angular momentum is conserved, $[H,\ J^i] = 0$, in the background of any spherically symmetric magnetic field $\vec{B}(\vec{x})$.

Let $\vec{J}$ be the orbital angular momentum, plus an improvement term that accounts for the angular momentum of the electromagnetic field

$$\vec{J} \;=\; \vec{x} \times \vec{p} - \vec{C}.$$

Then, we obtain the following conditions for $\vec{C}$

$$[x^i,\; C^j\;] \;=\; 0, \qquad [p^i,\; C^j\;] \;=\; ie\left(x^i B^j - \delta^{ij}(\vec{x}\cdot\vec{B})\right),$$

$$C^i \;=\; ex^i(\vec{x}\cdot\vec{B}) + \frac{i}{2}\,\epsilon^{ijk}\,[C^j,\; C^k\;].$$

The only consistent solution corresponds to the magnetic field of a Dirac monopole, in which case $J = \vec{x} \times \vec{p} - eg\;\hat{x}$ [Poincaré vector].

Non-associativity is responsible for the violation of angular symmetry.

The apparent violation of non-associativity in a Dirac monopole field is eliminated by imposing the boundary conditions $\Psi(0) = 0$ on the wave-functions so that $\vec{p}$ (derivations) are represented by self-adjoint operators, even though they are defined in patches as $\vec{p} = -i\nabla - e\vec{A}$.

Rotations by an angle θ around an axis $\hat{n}$ (take $\hat{n} = \hat{x}$) are described by

$$R(\hat{n} = \hat{x}, \theta) \;=\; e^{-i\theta\ \hat{x}\cdot\vec{J}} \;=\; e^{-ieg\ \theta}.$$

Then, for a point-particle in a monopole field, single valuedness of R (up to a sign) yields Dirac's quantization condition $eg = n \in \mathbb{Z} \quad (\times\hbar/2)$.
Finite translations in space also associate when eg is quantized.

In all other cases, non-associativity is for real, obstructing canonical quantization. What can be used as substitute? $\longrightarrow$ star-product.

DEFORMATIONS AND COHOMOLOGY

For definiteness we focus on the R-flux closed string model. Introduce (on dimensional grounds) the two physical constants, namely Planck's constant $\hbar$ and string length $l_s = \sqrt{\alpha'}$. We have

$$[x^i,\ x^j] \sim l_s^3 R \epsilon^{ijk} p_k, \qquad [x^i,\ p^j] \ = \ i\hbar\delta^{ij}, \qquad [p^i,\ p^j] \ = \ 0$$

whereas the dual coordinates and momenta satisfy the standard commutation relations

$$[\tilde{x}^i,\ \tilde{x}^j] \ = \ 0, \qquad [\tilde{x}^i,\ \tilde{p}^j] \ = \ i\hbar\delta^{ij}, \qquad [\tilde{p}^i,\ \tilde{p}^j] \ = \ 0.$$

Associator/Jacobiator does not vanish when either $\hbar$ or l_s are not zero.

$$[x^1,\ x^2,\ x^3] \ \equiv \ [[x^1,\ x^2],\ x^3] + \text{cycl. perm.} \sim \hbar l_s^3 R.$$

Different contractions of commutation relations of R-flux model:

$l_s = 0$, $\hbar = 0$: $\quad [x^i,\ x^j] = 0$, $\quad [x^i,\ p^j] = 0$ **(Algebra of translations $\mathfrak{t}_6$)**

$l_s = 0$, $\hbar \neq 0$: $\quad [x^i,\ x^j] = 0$, $\quad [x^i,\ p^j] = i\hbar\delta^{ij}$ **(Heisenberg algebra $\mathfrak{g}$)**

$l_s \neq 0$, $\hbar = 0$: $\quad [x^i,\ x^j] \sim l_s^3 \epsilon^{ijk} p^k$, $\quad [x^i,\ p^j] = 0$ **(3-central extension of $\mathfrak{t}_3$)**

whereas in all cases the momenta commute, $[p^i,\ p^j] = 0$.

The algebra of the R-flux model (and likewise that of the point-particle in $\vec{B} \sim \vec{x}$, letting $\vec{x} \leftrightarrow \vec{p}$) is deformation of the Lie algebras above.

Lie algebra cohomology characterizes the deformation that leads to non-associativity — Chevalley-Eilenberg cohomology

- **cochains of Abelian algebra $\mathfrak{t}_6$ with real values — $H^*(\mathfrak{t}_6,\ R)$**
- **cochains of Heisenberg algebra $\mathfrak{g}$ with values in g — $H^*(\mathfrak{g},\ \mathfrak{g})$**

$H^*(\mathbf{t}_6,\ R)$: **Let** $T_I = x^i, p^i$ **the generators of** $\mathbf{t}_6$**. Consider a 3-cochain with** $c_3(x^1, x^2, x^3) = 1$**, up to normalization, and** $c_3(T_I, T_J, T_K) = 0$ **for all other choices of generators (i.e., when at least one** T **is** p**). We have**

$$[T_I,\ T_J,\ T_K]\ \sim\ c_3(T_I, T_J, T_K)$$

and, thus, only the associator $[x^1, x^2, x^2]$ **does not vanish.**

The obstruction satisfies the 3-cocycle condition $dc_3(T_I, T_J, T_K, T_L) = 0$**, since for any four elements of** $\mathbf{t}_6$ **we have**

$$c_3([T_I, T_J],\ T_K,\ T_L) - c_3([T_I, T_K],\ T_J,\ T_L) + c_3([T_I, T_L],\ T_J,\ T_K) +$$

$$c_3([T_J, T_K],\ T_I,\ T_L) - c_3([T_J, T_L],\ T_I,\ T_K) + c_3([T_K, T_L],\ T_I,\ T_J)\ =\ 0$$

- c_3 **is not a coboundary, i.e.,** $c_3 \neq df_2$ **in** $H^*(\mathbf{t}_6,\ R)$**.**

$H^*(\mathbf{g},\ \mathbf{g})$**:** **Let** $c_2(x^i, x^j) = \epsilon^{ijk} p^k$**, up to a multiplicative constant, and** $c_2(x^i, p^j) = 0 = c_2(p^i, p^j)$**. Acting with the coboundary operator, we obtain**

$$dc_2(x^1, x^2, x^3) = -c_2([x^1,\ x^2],\ x^3) + c_2([x^1,\ x^3],\ x^2) - c_2([x^2,\ x^3],\ x^1)$$

$$+\pi(x^1)c_2(x^2, x^3) - \pi(x^2)c_2(x^1, x^3) + \pi(x^3)c_2(x^1, x^2)$$

where $\pi(\mathbf{g}) = \mathrm{Ad}_{\mathbf{g}} = [\mathbf{g},\ \cdot\ ]$**. Then, for the Heisenberg algebra, we have**

$$dc_2(x^1, x^2, x^3) = [x^1,\ c_2(x^2, x^3)] - [x^2,\ c_2(x^1, x^3)] + [x^3,\ c_2(x^1, x^2)]$$

leading to alternative cohomological interpretation of non-associativity

$$[x^1,\ x^2,\ x^3] \sim dc_2(x^1, x^2, x^3)$$

- **The cohomological interpretation depends on the module** $(R$ **vs** $\mathbf{g})$**.**

LIE GROUP COHOMOLOGY

Exponentiate the action of the position and momentum generators. The formal group elements

$$U(\vec{a},\ \vec{b}) \ = \ e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p})}$$

satisfy the composition law, obtained by applying BCH formula,

$$U(\vec{a}_1,\vec{b}_1)U(\vec{a}_2,\vec{b}_2) \ = \ e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2-\vec{a}_2\cdot\vec{b}_1)}\ e^{-i\frac{R}{2}(\vec{a}_1\times\vec{a}_2)\cdot\vec{p}}\ U(\vec{a}_1+\vec{a}_2,\ \vec{b}_1+\vec{b}_2).$$

Successive composition of any three group elements $U_i = U(\vec{a}_i,\vec{b}_i)$ **yields**

$$(U_1\ U_2)\ U_3 \ = \ e^{-i\frac{R}{2}(\vec{a}_1\times\vec{a}_2)\cdot\vec{a}_3}\ U_1\ (U_2\ U_3)$$

which do not associate when $R\neq 0$**.**

If R were zero, we would have a projective representation of the Abelian group of translations in phase space. The phase factor

$$\varphi_2(\vec{a}_1, \vec{b}_1; \vec{a}_2, \vec{b}_2) = \vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1$$

is a real-valued 2-cocycle in group cohomology, satisfying

$$d\varphi_2(\vec{a}_1, \vec{a}_2, \vec{a}_3) \equiv \varphi_2(\vec{a}_2, \vec{a}_3) - \varphi_2(\vec{a}_1 + \vec{a}_2, \vec{a}_3) + \varphi_2(\vec{a}_1, \vec{a}_2 + \vec{a}_3) - \varphi_2(\vec{a}_1, \vec{a}_2) = 0.$$

If $R \neq 0$, there is an additional p-dependent factor in the composition law that gives rise to a phase in the associator of three group elements

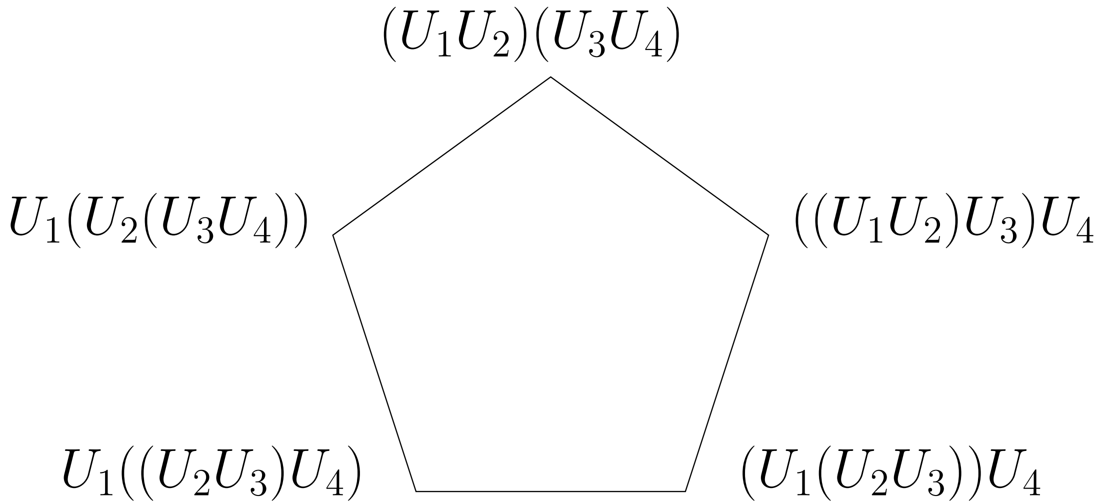
$$\varphi_3(\vec{a}_1, \vec{a}_2, \vec{a}_2) = (\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3$$

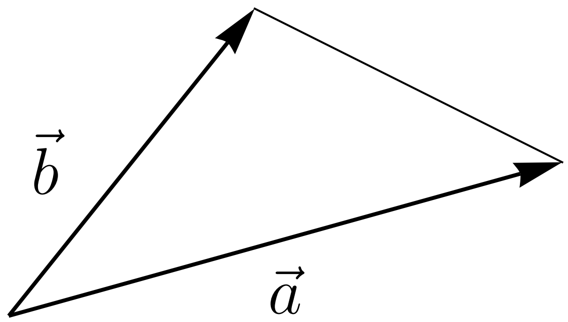
- **φ_2 does not show up in the associator because $d\varphi_2 = 0$.**

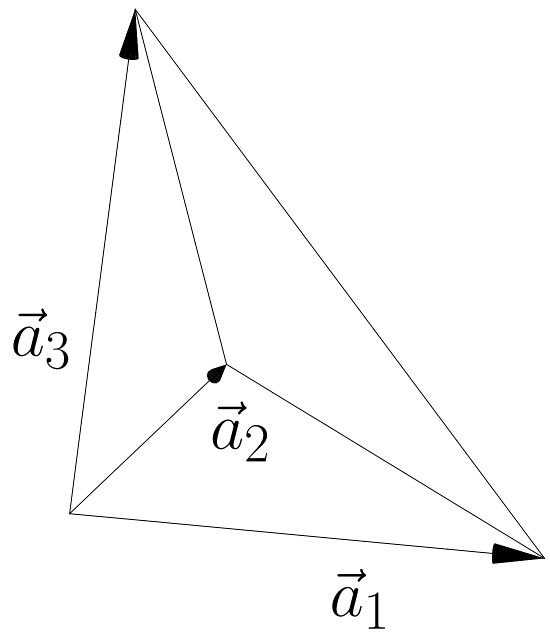
The new phase is a real-valued 3-cocycle in the cohomology of the Abelian group of translations in phase space, satisfying

$$d\varphi_3(\vec{a}_1,\vec{a}_2,\vec{a}_3,\vec{a}_4) \;\equiv\; \varphi_3(\vec{a}_2,\vec{a}_3,\vec{a}_4) - \varphi_3(\vec{a}_1+\vec{a}_2,\vec{a}_3,\vec{a}_4) +$$
$$\varphi_3(\vec{a}_1,\vec{a}_2+\vec{a}_3,\vec{a}_4) - \varphi_3(\vec{a}_1,\vec{a}_2,\vec{a}_3+\vec{a}_4) + \varphi_3(\vec{a}_1,\vec{a}_2,\vec{a}_2) \;=\; 0 \; .$$

A schematic representation is provided by Mac Lane's pentagon:







Geometric interpretation of the non-trivial cocycles φ_2 and φ_3:

$$\mathrm{Area}(\vec{a},\vec{b}) = \frac{1}{2}\ |\vec{a}\times\vec{b}|$$

$$\mathrm{Volume}(\vec{a}_1,\vec{a}_2,\vec{a}_3) = \frac{1}{6}\ |(\vec{a}_1\times\vec{a}_2)\cdot\vec{a}_3|$$

ALTERNATIVE INTERPRETATION of non-associativity is provided by the cohomology of Heisenberg-Weyl group with cochains taking values in the Heisenberg algebra. Introducing the elements

$$U_W(g) \ = \ e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p}+c\mathbf{1})}$$

the group composition law $U(\vec{a}_1, \vec{b}_1)U(\vec{a}_1, \vec{b}_1)$ takes the form

$$U_W(g_1)\ U_W(g_2) \ = \ e^{-i\frac{R}{2}\varphi_2(g_1,g_2)}\ U_W(g_1 g_2)$$

where $\varphi_2(g_1, g_2) = (\vec{a}_1 \times \vec{a}_2) \cdot \vec{p}$ takes values in the Heisenberg algebra. Then, the obstruction to associativity assumes the coboundary form

$$(\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3 = d\varphi_2(g_1, g_2, g_3).$$

As before, the cohomological interpretation depends on the module.

THE STAR PRODUCT

When $R = 0$, all classical observables $f(x,p)$ are assigned to operators $\hat{F}(\hat{x},\hat{p})$ acting on Hilbert space $\mathcal{H}$. Their product is non-commutative but associative.

An equivalent description is provided by Moyal star-product in phase space: Fourier analyse

$$f(\vec{x},\vec{p}) = \frac{1}{(2\pi)^3}\int d^3a d^3b \; \tilde{f}(\vec{a},\vec{b}) e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p})}$$

and apply Weyl's correspondence rule to assign self-adjoint operators

$$\hat{F}(\hat{\vec{x}},\hat{\vec{p}}) = \frac{1}{(2\pi)^3}\int d^3a d^3b \; \tilde{f}(\vec{a},\vec{b}) \hat{U}(\vec{a},\vec{b})$$

where

$$\hat{U}(\vec{a},\ \vec{b}) = e^{i(\vec{a}\cdot\hat{\vec{x}}+\vec{b}\cdot\hat{\vec{p}})}.$$

The product of any two operators takes the form

$$\hat{F}_1\cdot\hat{F}_2 = \frac{1}{(2\pi)^6}\int d^3a_1 d^3b_1 d^3a_2 d^3b_2\ \tilde{f}_1(\vec{a}_1,\vec{b}_1)\tilde{f}_2(\vec{a}_2,\vec{b}_2)\hat{U}(\vec{a}_1,\ \vec{b}_1)\hat{U}(\vec{a}_2,\ \vec{b}_2)$$

and it can be worked out using the composition law

$$\hat{U}(\vec{a}_1,\ \vec{b}_1)\hat{U}(\vec{a}_2,\ \vec{b}_2) = e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2-\vec{a}_2\cdot\vec{b}_1)}\hat{U}(\vec{a}_1+\vec{a}_2,\ \vec{b}_1+\vec{b}_2).$$

The 2-cocycle $\varphi_2(\vec{a}_1,\vec{b}_1;\vec{a}_1,\vec{b}_1) = \vec{a}_1\cdot\vec{b}_2 - \vec{a}_2\cdot\vec{b}_1$ **makes the product of the corresponding phase space functions non-commutative but associative. The result turns out to be**

$$(f_1 \star f_2)(\vec{x}, \vec{p}) = e^{\frac{i}{2}\left(\vec{\nabla}_{x_1} \cdot \vec{\nabla}_{p_2} - \vec{\nabla}_{x_2} \cdot \vec{\nabla}_{p_1}\right)} f_1(\vec{x}_1, \vec{p}_1) f_2(\vec{x}_2, \vec{p}_2)|_{\vec{x}_1 = \vec{x}_2 = \vec{x};\ \vec{p}_1 = \vec{p}_2 = \vec{p}}$$

giving rise to the series expansion

$$(f_1 \star f_2)(\vec{x}, \vec{p}) = (f_1 \cdot f_2)(\vec{x}, \vec{p}) + \frac{i}{2}\{f_1,\ f_2\} + \cdots\ .$$

Non-commutative geometry: the notion of point becomes fuzzy.

Quantum dynamics is equivalently described by the Moyal bracket

$$\{\{f_1,\ f_2\}\} \equiv -i(f_1 \star f_2 - f_2 \star f_1) = \{f_1,\ f_2\} + \text{higher derivatives}$$

acting as derivation

$$\{\{f_1,\ f_2 \star f_3\}\} = f_2 \star \{\{f_1,\ f_3\}\} + \{\{f_1,\ f_2\}\} \star f_3.$$

When $R \neq 0$, the rules of canonical quantization do not apply, but it is still possible to define a star-product non-commutative/non-associative.

We follow the same line of thought as before, assigning to $f(\vec{x}, \vec{p})$

$$F(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \; \tilde{f}(\vec{a}, \vec{b}) U(\vec{a}, \vec{b}),$$

and using the generalized composition law,

$$U(\vec{a}_1, \vec{b}_1) U(\vec{a}_2, \vec{b}_2) \;=\; e^{-\frac{i}{2}(\vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1)} \, e^{-i\frac{R}{2}(\vec{a}_1 \times \vec{a}_2) \cdot \vec{p}} \; U(\vec{a}_1 + \vec{a}_2, \; \vec{b}_1 + \vec{b}_2).$$

The result is the NC/NA p-dependent star-product

$$(f_1 \star_p f_2)(\vec{x}, \vec{p}) = e^{i\frac{R}{2} \; \vec{p} \cdot (\vec{\nabla}_{x_1} \times \vec{\nabla}_{x_2})} e^{\frac{i}{2}\left(\vec{\nabla}_{x_1} \cdot \vec{\nabla}_{p_2} - \vec{\nabla}_{x_2} \cdot \vec{\nabla}_{p_1}\right)}$$

$$f_1(\vec{x}_1, \vec{p}_1) f_2(\vec{x}_2, \vec{p}_2)|_{\vec{x}_1 = \vec{x}_2 = \vec{x}; \; \vec{p}_1 = \vec{p}_2 = \vec{p}} \; .$$

The substitute for quantum dynamics is provided by the bracket

$$\{\{f_1,\ f_2\}\}_p \equiv -i(f_1 \star_p f_2 - f_2 \star_p f_1)$$

which does not act as derivation, i.e.,

$$\{\{f_1,\ f_2 \star_p f_3\}\}_p\ \neq\ f_2 \star_p \{\{f_1,\ f_3\}\}_p + \{\{f_1,\ f_2\}\}_p \star_p f_3.$$

A related result is that the associator/Jacobiator does not vanish

$$\{\{f_1(x),\ f_2(x),\ f_3(x)\}\}_p\ \neq\ 0.$$

- **The star-product extends naturally to double phase space** $(x, p; \tilde{x}, \tilde{p})$ **by combining** $\star_p$**-product in** (x, p) **with Moyal product in** $(\tilde{x}, \tilde{p})$**.**
- **Symmetries appear to be broken as consequence of non-associativity.**

SUMMARY/CONCLUSIONS/QUESTIONS

- Non-geometric closed string backgrounds exhibit NC/NA
- T-folds, double field theory, generalized geometry should be advanced further to accommodate old and new structure on equal footing
- Exploit deformation theory of Lie algebras in relation to deformation theory of complex structures
- Systematic generalization to other backgrounds (e.g., with elliptic monodromies)
- Relevance of NC/NA to coordinate dependent compactification of string theory and gauged supergravities

THANK YOU!