Membrane σ-models and quantization of non-geometric flux backgrounds

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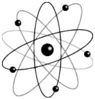
joint work with D. Mylonas and R.J. Szabo
JHEP 1209, 012 (2012)

Introduction/Motivation

Planck scale quantum geometry

Heuristic argument: quantum + gravity



"The gravitational field generated by the concentration of energy required to localize an event in spacetime should not be so strong as to hide the event itself to a distant observer."

- fundamental length scale, spacetime uncertainty $\Delta x \geq l_P$
- uncertainty principle $\stackrel{?}{\longleftrightarrow}$ noncommutative spacetime structure

Strings and noncommutative geometry

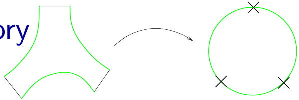
- D-brane + B-field $\rightarrow$ noncommutative space
- noncommutative gauge theory, Seiberg-Witten maps
- NC Standard Model, GUTs, bundles and gerbes

Chu, Ho (1998), Schomerus (1999), Seiberg, Witten (1999)
Jurco, Madore, Schraml, PS, Wess (2000) + many more

Introduction/Motivation

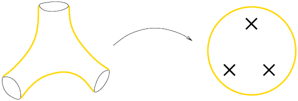
in closed string background. . .

open string noncommutative gauge theory



- 2+1 points on boundary of disk: ordering
- 2-tensor (B-field) $\Rightarrow$ noncommutative 2-bracket
- quantization: $\star$-product

closed string nonassociative gravity?



- 3+1 points on sphere: orientation
- 3-tensor $\Rightarrow$ nonassociative 3-bracket

Blumenhagen, Plauschinn (2010), Lüst (2010)

- quantization? (yes: dynamical $\star$-product)
- SW map? (yes: several types)
- non-constant backgrounds? (possible)

Introduction/Motivation

Previously: *nonassociative gauge theory* and *noncommutative gravity*

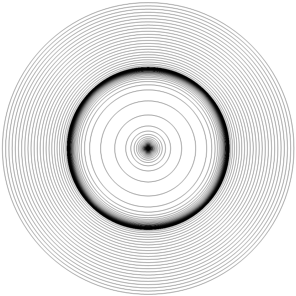
- NC gauge theory with 3-form $H = dB \neq 0 \;\Rightarrow$ nonassociative spacetime; locally described by associative algebras & NC gerbes

  Cornalbe, Schiappa (2002); Aschieri, Bakovic, Jurco, PS (2010)

- Noncommutative gravity, General Relativity on noncommutative spacetime; exact solutions: fuzzy black holes

  Aschieri, Blohmann, Dimitrijevic, Meyer, PS, Wess (2005)

  Solodukhin, PS (2009)



Flux compactification

Relating string theory to observably phenomenology and cosmology requires compactification. Fluxes stabilize moduli and can lead to generalized geometric structures; patching by string symmetries.

Non-geometric flux backgrounds

T-dualizing a 3-torus with 3-form H-flux gives rise to geometric and non-geometric fluxes $\quad H_{abc} \longrightarrow f^a{}_{bc} \longrightarrow Q^{ab}{}_c \longrightarrow R^{abc}$

Hull (2005)

Shelton, Taylor, Wecht (2005)

Q-flux: momentum and winding modes mix $\rightarrow$ T-folds
R-flux: only beginning to understand its (non)geometry
$\rightarrow$ non-commutative non-associative structures

Lüst (2010), Blumenhagen, Plauschinn (2010)

Blumenhagen, Deser, Lüst, Plauschinn, Rennecke (2011)

Outline

- AKSZ sigma-models for geometric and non-geometric backgrounds
- Quantization $\Rightarrow$ non-commutative non-associative geometry
- Seiberg-Witten-type maps, dynamical star product
- Remarks on Nambu-Poisson structures

AKSZ sigma-models

AKSZ construction: action functionals in BV formalism of sigma model QFT's for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) \ ,$$

with $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$, $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$

perturbative expansion $\Rightarrow$ Kontsevich formality maps

(valid on-shell ($[\Theta,\Theta]_S = 0$) as well as off-shell, e.g. twisted Poisson)

Courant sigma model

TFT with 3-dimensional membrane world volume Σ_3

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, h_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - P_I{}^i(X)\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K\Big)$$

with embeddings $X : \Sigma_3 \to M$, 1-form α, aux. 2-form ϕ, fibre metric h, anchor matrix P, 3-form T.

standard Courant algebroid:
$C = TM \oplus T^*M$ with natural frame (ϱ_i, χ^i), metric $\langle \varrho_i, \chi^j\rangle = \delta_i{}^j$

H-space sigma-model

relevant for geometric flux compactifications: $C = TM \oplus T^*M$ twisted by 3-form flux $H = \frac{1}{6}\, H_{ijk}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$

H-twisted Courant–Dorfman bracket

$$\big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_H := \big([Y_1,Y_2]_{TM}\,,\,\mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1)-\alpha_1(Y_2)\big) + H(Y_1,Y_2,-)\big)$$

metric: natural dual pairing

$$\big\langle(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big\rangle = \alpha_2(Y_1)+\alpha_1(Y_2)$$

anchor map: projection $\rho : C \to TM$
non-trivial bracket and 3-bracket

$$[\varrho_i,\varrho_j]_H = H_{ijk}\,\chi^k\ , \qquad [\varrho_i,\varrho_j,\varrho_k]_H = H_{ijk}$$

H-space sigma-model action

$$S^{(2)}_{\mathrm{WZ}} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \alpha^i \wedge \mathrm{d}\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6}\, H_{ijk}(X)\, \alpha^i \wedge \alpha^j \wedge \alpha^k \Big) \ .$$

where $(\alpha^I) = (\alpha^1, \dots, \alpha^{2d}) \equiv (\alpha^1, \dots, \alpha^d, \xi_1, \dots, \xi_d)$

If $\Sigma_2 := \partial\Sigma_3 \neq \emptyset$, we can add a boundary term $\Rightarrow$
boundary/bulk open topological membrane action

$$\widetilde{S}^{(2)}_{\mathrm{WZ}} = S^{(2)}_{\mathrm{WZ}} + \int_{\Sigma_2} \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j \ .$$

(other boundary terms are possible, but will not be considered here)

H-space sigma-model

H-twisted Poisson sigma-model

Integrating out the two-form fields ϕ_i yields the AKSZ action

$$\widetilde{S}^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) + \int_{\Sigma_3} \frac{1}{6}\,H_{ijk}(X)\,\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ ,$$

which is the action of the H-twisted Poisson sigma-model with target space M. Consistency of the equations of motion require Θ to be H-twisted Poisson, i.e.

$$[\Theta,\Theta]_{\mathrm{S}} = \textstyle\bigwedge^3 \Theta^\sharp(H) \neq 0$$

$\Rightarrow$ the Jacobi identity for the bracket is violated.

R-space sigma-model

From H to Q to R

Closed strings in Q-space via two T-duality transformations on 3-torus $\mathbb{T}^3$; locally filtration of $\mathbb{T}^2$ over S^1, globally not well-defined (T-fold). Closed string world sheet $\mathcal{C} = \mathbb{R} \times S^1$, coordinates (σ^0, σ^1), winding number $\tilde{p}^3$, twisted boundary conditions at σ'^1.

Closed string non-commutativity expressed via Poisson brackets:

$$\{x^i, x^j\}_Q = Q^{ij}{}_k\, \tilde{p}^{\,k} \qquad \text{and} \qquad \{x^i, \tilde{p}^{\,j}\}_Q = 0 = \{\tilde{p}^{\,i}, \tilde{p}^{\,j}\}_Q$$

Another T-duality transformation sends $Q^{ij}{}_k \mapsto R^{ijk}$, $\tilde{p}^k \mapsto p_k$ and the Poisson brackets to the twisted Poisson structure

$$\{x^i, x^j\}_\Theta = R^{ijk}\, p_k \ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{p_i, p_j\}_\Theta = 0 \ .$$

Lüst (2010,2012)

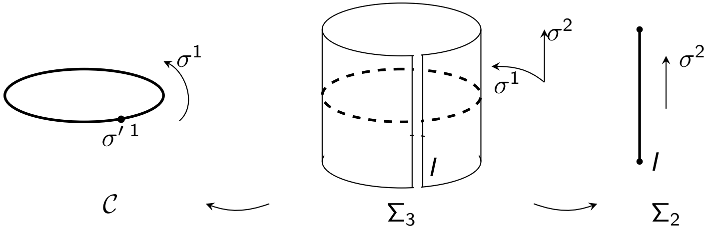
The hidden open string

CFT computation: insert twist field at $\sigma'^1 \in S^1$ $\rightarrow$ generates branch cut

There are indications that the appropriate R-space theory is a membrane sigma model, not a string theory:

- open strings do not decouple from gravity in R-space
- membrane theory geometrizes the non-geometric R-flux background

$\Rightarrow$ extend world sheet $\mathcal{C}$ to membrane world volume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$; resulting branch surface can be interpreted as open string world sheet:



closed $\leftrightarrow$ open string duality

R-space sigma-model

General Courant sigma-model with standard Courant algebroid $C = TM \oplus T^*M$, twisted by a trivector flux $R = \frac{1}{6} R^{ijk}(x)\,\partial_i \wedge \partial_j \wedge \partial_k$.

Roytenberg's R-twisted Courant-Dorfman bracket

$$\big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_R := \big([Y_1,Y_2]_{TM} + R(\alpha_1,\alpha_2,-)\,,\\ \mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1) - \alpha_1(Y_2)\big)\big)$$

non-trivial bracket and 3-bracket

$$[\chi^i,\chi^j]_R = R^{ijk}\,\varrho_k\ , \qquad [\chi^i,\chi^j,\chi^k]_R = R^{ijk}\ .$$

R-space sigma-model

R-space sigma-model action

$$S_R^{(2)} = \int_{\Sigma_3} \Big(\phi_i \wedge \big(\mathrm{d}X^i - \alpha^i\big) + \alpha^i \wedge \mathrm{d}\xi_i + \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k\Big) + \frac{1}{2} \int_{\Sigma_2} g^{ij}(X)\, \xi_i \wedge *\xi_j \ ,$$

where we have added a non-topological term involving g^{ij}, to ensure consistency of $R^{ijk} \neq 0$.

Integrating out the 2-form field ϕ yields:

$$\boxed{S_R^{(2)} = \int_{\Sigma_2} \xi_i \wedge \mathrm{d}X^i + \int_{\Sigma_3} \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}(X)\, \xi_i \wedge *\xi_j \ .}$$

assume now constant R^{ijk} and g^{ij} and consider e.o.m. for X ...

R-space sigma-model

$\Rightarrow \xi_i = dP_i$ and the action reduces to a pure boundary action:

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\mathrm{d}P_i \wedge \mathrm{d}X^i + \frac{1}{2}\, R^{ijk}\, P_i\, \mathrm{d}P_j \wedge \mathrm{d}P_k \Big) + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}\, \mathrm{d}P_i \wedge *\mathrm{d}P_j \ ,$$

which can be rewritten as

$$S_R^{(2)} = \int_{\Sigma_2} -\frac{1}{2}\, \Theta^{-1}_{IJ}(X)\, \mathrm{d}X^I \wedge \mathrm{d}X^J + \int_{\Sigma_2} \frac{1}{2}\, g_{IJ}\, \mathrm{d}X^I \wedge *\mathrm{d}X^J \ ,$$

with

$$\Theta^{-1} = \big(\Theta^{-1}_{IJ}\big) = \begin{pmatrix} 0 & -\delta_i{}^j \\ \delta^i{}_j & R^{ijk}\, p_k \end{pmatrix} \ , \qquad \big(g_{IJ}\big) = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix}$$

and $X = (X^I) = (X^1, \dots, X^{2d}) := (X^1, \dots, X^d, P_1, \dots, P_d)$.

$\Rightarrow$ effective target space = phase space

The “closed string metric” g_{IJ} acts only on momentum space.

R-space sigma-model

Linearized action

Generalized Poisson sigma-model

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\, \Theta^{IJ}(X)\, \eta_I \wedge \eta_J \Big) + \int_{\Sigma_2} \frac{1}{2}\, G^{IJ}\, \eta_I \wedge *\eta_J \ ,$$

with auxiliary fields η_I and

$$\Theta = \big(\Theta^{IJ}\big) = \begin{pmatrix} R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \ , \qquad \big(G^{IJ}\big) = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

obeying the usual closed-open string relations, w.r.t. Θ^{-1} and g.

In phase-space component form:

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_i \wedge \mathrm{d}X^i + \pi^i \wedge \mathrm{d}P_i + \frac{1}{2}\, R^{ijk}\, P_k\, \eta_i \wedge \eta_j + \eta_i \wedge \pi^i \Big) + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}\, \eta_i \wedge *\eta_j \ ,$$

with $(\eta_I) = (\eta_1, \ldots, \eta_{2d}) \equiv (\eta_1, \ldots, \eta_d, \pi^1, \ldots, \pi^d)$.

R-space sigma-model

Non-commutative, non-associative phase space

Θ is an H-twisted Poisson bi-vector: $[\Theta,\Theta]_{\rm S} = \bigwedge^3 \Theta^\sharp(H)$, where

$$H = \tfrac{1}{6}\, R^{ijk}\, \mathrm{d}p_i \wedge \mathrm{d}p_j \wedge \mathrm{d}p_k = dB \ , \ \text{and} \ \ B = \tfrac{1}{6}\, R^{ijk}\, p_k\, \mathrm{d}p_i \wedge \mathrm{d}p_j \ .$$

Twisted Poisson brackets

$$\{x^i, x^j\}_\Theta = R^{ijk}\, p_k \ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{p_i, p_j\}_\Theta = 0 \ .$$

Corresponding Jacobiator:

$$\{x^i, x^j, x^k\}_\Theta = R^{ijk} \ ,$$

where $\{x^I, x^J, x^K\}_\Theta := [\Theta,\Theta]_{\rm S}(x^I, x^J, x^K) = \Pi^{IJK}$ and

$$\big(\Pi^{IJK}\big) = \frac{1}{3}\,\big(\Theta^{KL}\,\partial_L\Theta^{IJ} + \Theta^{IL}\,\partial_L\Theta^{JK} + \Theta^{JL}\,\partial_L\Theta^{KI}\big) = \begin{pmatrix} R^{ijk} & 0 \\ 0 & 0 \end{pmatrix} \ .$$

Path integral quantization

Mapping the open string endpoints to finite values and imposing natural boundary conditions, we are let to the following schematic functional integrals that reproduce Kontsevich's graphical expansion of global deformation quantization. For multivector fields $\mathcal{X}_r$ of degree k_r:

$$U_n(\mathcal{X}_1,\dots,\mathcal{X}_n)(f_1,\dots,f_m)(x) = \int \mathrm{e}^{\,\frac{\mathrm{i}}{\hbar}\, S_R^{(2)}}\, S_{\mathcal{X}_1}\cdots S_{\mathcal{X}_n}\,\mathcal{O}_x(f_1,\dots,f_m)\,,$$

where $m = 2 - 2n + \sum_r k_r$, $S_{\mathcal{X}_r} = \frac{\mathrm{i}}{\hbar}\int_{\Sigma_2}\frac{1}{k_r!}\,\mathcal{X}_r^{I_1\dots I_r}(X)\,\eta_{I_1}\cdots\eta_{I_r}$, and

$$\mathcal{O}_x(f_1,\dots,f_m) = \int_{X(\infty)=x}\Big[f_1\big(X(q_1)\big)\cdots f_m\big(X(q_m)\big)\Big]^{(m-2)}\,,$$

with $1 = q_1 > q_2 > \cdots > q_m = 0$ and ∞ distinct points on the boundary of the disk $\partial\Sigma_2$; the path integrals are weighted with the full gauge-fixed action and the integrations taken over all fields including ghosts.

Cattaneo, Felder (2000)

Quantization

Kontsevich formality maps

U_n maps n multivector fields to a differential operator

$$U_n(\mathcal{X}_1, \dots, \mathcal{X}_n) = \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \dots, \mathcal{X}_n) \ ,$$

where the sum is over all possible diagrams with weight

$$w_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_i^1} \wedge \cdots \wedge \mathrm{d}\phi^h_{e_i^{k_i}} \right) \ .$$

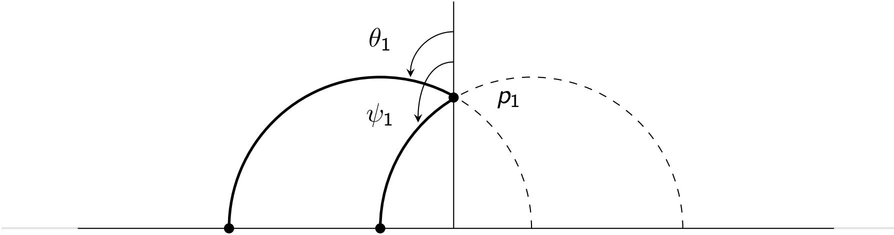
The star product and the 3-bracket are given by

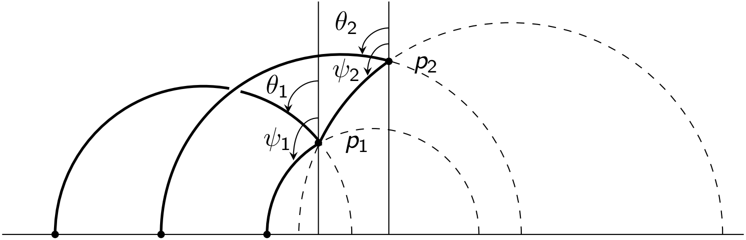
$$f \star g = \sum_{n=0}^{\infty} \frac{(\,\mathrm{i}\,\hbar)^n}{n!} \, U_n(\Theta, \dots, \Theta)(f, g) =: \Phi(\Theta)(f, g) \ ,$$

$$[f, g, h]_\star = \sum_{n=0}^{\infty} \frac{(\,\mathrm{i}\,\hbar)^n}{n!} \, U_{n+1}(\Pi, \Theta, \dots, \Theta)(f, g, h) =: \Phi(\Pi)(f, g, h) \ .$$

Quantization

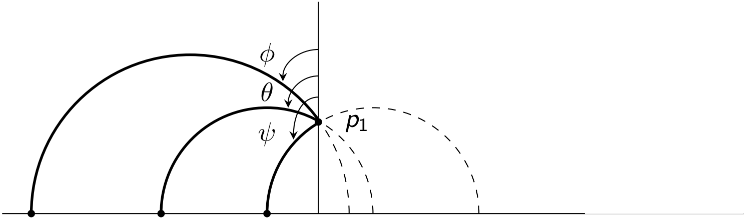
Relevant diagrams involve the bivector $\Theta = \frac{1}{2}\Theta^{IJ}\partial_I \wedge \partial_J \ldots$





Quantization

. . . and the trivector $\Pi = \frac{1}{6}\Pi^{IJK}\partial_I \wedge \partial_J \wedge \partial_K = d_\Theta\Theta = [\Theta, \Theta]_S$:



For constant Π all other diagrams factorize and their weights can be expressed in terms of these three diagrams (up to permutations).

Formality condition

The U_n define L_∞-morphisms and satisfy

$$\mathrm{d}.\,U_n(\mathcal{X}_1,\dots,\mathcal{X}_n)+\frac{1}{2}\sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\dots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}}\varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J})\,\big[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}})\,,\,U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\big]_{\mathrm{G}}$$
$$=\ \sum_{i<j}(-1)^{\alpha_{ij}}\,U_{n-1}\big([\mathcal{X}_i,\mathcal{X}_j]_{\mathrm{S}},\mathcal{X}_1,\dots,\widehat{\mathcal{X}}_i,\dots,\widehat{\mathcal{X}}_j,\dots,\mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

Kontsevich (1997)

This implies in particular

$$\mathrm{d}_\star\Phi(\Theta)=\ \mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\Theta\Theta)\ ,$$

which explicitly quantifies the lack of associativity of the star product:

$$(f\star g)\star h-f\star(g\star h)=\tfrac{\hbar}{2\,\mathrm{i}}[f,g,h]_\star=\tfrac{\hbar}{2\,\mathrm{i}}\Phi(\Pi)(f,g,h)\ .$$

Quantization

The formality condition implies derivation properties:

- For a function h, the Hamiltonian vector field $\mathrm{d}_\Theta h = \{-, h\}$ is mapped to the inner derivation $\mathrm{d}_\star \underline{h} = \frac{\mathrm{i}}{\hbar}\,[\,\underline{h}\,, -]_\star = i\hbar\Phi(\mathrm{d}_\Theta h)$, where $\underline{h} = \Phi(h) \equiv \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_{n+1}(h, \Theta, \dots, \Theta)$.
- A Poisson structure preserving vector field $\mathcal{X}$ ($\mathrm{d}_\Theta \mathcal{X} = 0$) is mapped to a differential operator $\underline{\mathcal{X}} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_{n+1}(\mathcal{X}, \Theta, \dots, \Theta)$ satisfying $\mathcal{X}(f \star g) = \mathcal{X}(f) \star g + f \star \mathcal{X}(g)$.
- The formality condition $\mathrm{d}_\star \Phi(\Pi) = i\hbar\Phi(\mathrm{d}_\Theta \Pi)$ and higher derivation properties encode quantum analogs of the derivation property and fundamental identity for a Nambu-Poisson structure.
- In particular, in the present case, where $\mathrm{d}_\Theta \Pi = 0$:

$$[f \star g, h, k]_\star - [f, g \star h, k]_\star + [f, g, h \star k]_\star = f \star [g, h, k]_\star + [f, g, h]_\star \star k\ .$$

Explicit formulas

- Dynamical non-associative star product: $f \star g \equiv f \star_p g$, with

$$f \star_p g = \cdot \Big[e^{\frac{\mathrm{i}\,\hbar}{2}\, R^{ijk}\, p_k\, \partial_i \otimes \partial_j}\; e^{\frac{\mathrm{i}\,\hbar}{2}\, \left(\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right)} (f \otimes g) \Big]$$

- Replacing the dynamical variable p with a constant $\tilde{p}$ we obtain an associative Moyal-Weyl type star product $\tilde{\star} := \star_{\tilde{p}}$.
- Triple products and 3-bracket:

$$(f \star g) \star h = \Big[\, \tilde{\star} \Big(\exp\left(\tfrac{\hbar^2}{4}\, R^{ijk}\, \partial_i \otimes \partial_j \otimes \partial_k\right)(f \otimes g \otimes h)\Big) \Big]_{\tilde{p} \to p}$$

$$[f, g, h]_\star = \frac{4\,\mathrm{i}}{\hbar} \Big[\, \tilde{\star} \Big(\sinh\left(\tfrac{\hbar^2}{4}\, R^{ijk}\, \partial_i \otimes \partial_j \otimes \partial_k\right)(f \otimes g \otimes h)\Big) \Big]_{\tilde{p} \to p}$$

- Trace property: $\int [f, g, h]_\star = 0$

Seiberg-Witten map

The map from ordinary to NC gauge theory is related to the equivalence map $\mathcal{D}$ of star products $\star$, $\star'$ and is a quantum analog of Moser's lemma.

Let $F = dA$ and ρ the flow generated by the vector field $A_\Theta = \Theta(A, -)$:

$$
\begin{array}{ccccc}
B: & & \Theta & \xrightarrow{\text{quantization}} & \star \\
\text{Moser}\ \big\downarrow \rho & & \big\downarrow \rho & & \big\downarrow \mathcal{D} \\
B + F: & & \Theta' & \xrightarrow{\text{quantization}} & \star'
\end{array}
$$

where $\Theta' = \Theta(1 + \hbar F \Theta)^{-1}$ and $\mathcal{D}(f \star' g) = \mathcal{D}f \star \mathcal{D}g$.

The noncommutative gauge field $\hat{A}$ is obtained from $\mathcal{D}x =: x + \hat{A}$, such that ordinary gauge transform of A $\Rightarrow$ NC gauge transform of $\hat{A}$.

$\rightarrow$ explicit expression for the SW map for arbitrary $\Theta(x)$

$\rightarrow$ can be globalized (and extended to gerbes)

Jurco, PS, Wess (2000-2002)

Seiberg-Witten maps

Twisted Poisson structure, NC gerbes

Poisson structure twisted by closed 3-form H: $[\Theta, \Theta]_S = \bigwedge^3 \Theta^\sharp H$

For covering by contractible open patches labeled by $\alpha, \beta, \gamma, \ldots$:

$$H|_\alpha = dB_\alpha \ , \qquad (B_\beta - B_\alpha)|_{\alpha\cap\beta} = F_{\alpha\beta} = dA_{\alpha\beta}$$

Θ can be locally untwisted by B_α: $\Theta_\alpha := \Theta(1 - \hbar B_\alpha \Theta)^{-1}$.

quantization of Θ $\rightarrow$ nonassociative $\star$
quantization of Θ_α, Θ_β $\rightarrow$ associative $\star_\alpha$, $\star_\beta$ related by $\mathcal{D}_{\alpha\beta}$

for more details: Aschieri, Bakovic, Jurco, PS (2010)

SW maps for R-twisted Poisson structures

trivial gerbe $\rightarrow$ replace patch label α by the (constant) vector $\tilde{p}$:

$$\Theta = \begin{pmatrix} \hbar\, R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \Theta_{\tilde{p}} = \begin{pmatrix} \hbar\, R^{ijk}\, \tilde{p}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad B_{\tilde{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk}\,(p_k - \tilde{p}_k) \end{pmatrix}$$

Θ: twisted Poisson $\qquad$ $\Theta_{\tilde{p}}$: Poisson $\qquad$ $H = dB_{\tilde{p}} = \frac{1}{2} R^{ijk} dp_i dp_j dp_k$

Seiberg-Witten maps

Gauge potential: $A = A_I \mathrm{d}x^I = a_i(x,p)\mathrm{d}x^i + \tilde{a}^i(x,p)\mathrm{d}p_i$

Maps between *associative* $\tilde{\star}$ and $\tilde{\star}'$ are generated by $A_{\tilde{p}\tilde{p}'} = R^{ijk} p_i(\tilde{p}_k - \tilde{p}'_k)\mathrm{d}p_j$ with $F_{\tilde{p}\tilde{p}'} = R^{ijk}(\tilde{p}_k - \tilde{p}'_k)\mathrm{d}p_i\mathrm{d}p_j$.

Special case $\tilde{p} = 0$: canonical Moyal-Weyl star product $\star_0$.

Generalization of SW maps for non-associative structures

A construction directly based on twisted Θ is spoiled by $[\Theta, \Theta]_S$-terms. These can be avoided in the present case by choosing $a_i(x,p) = 0$!

- general coordinate transformations generated by $\Theta(A, -) = \tilde{a}^i(x,p)\partial_i$
- Nambu-Poisson maps: choose $A = R(a_2, -)$ for *any* 2-form a_2; $\rightarrow$ higher "Nambu-Poisson" gauge theory.
- map from associative to nonassociative: $\mathcal{D}_{\tilde{p}}$ generated by $A_{\tilde{p}} = \frac{1}{2} R^{ijk} p_i \tilde{p}_k \mathrm{d}p_j$ can be explicitly computed and satisfies

$$f \star g = \left[\mathcal{D}_{\tilde{p}} f \star_0 \mathcal{D}_{\tilde{p}} g\right]_{\tilde{p} \to p}$$

Nambu-Poisson

Remarks on Nambu-Poisson structures

- The trivector $\Pi = \frac{1}{6} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$ is an example of a Nambu-Poisson tensor. More generally:

$$\{f, h_1, \ldots, h_p\} = \Pi^{i\, j_1 \ldots j_p}(x)\, \partial_i f\, \partial_{j_1} h_1 \, \cdots\, \partial_{j_p} h_p$$

$$\begin{aligned} \{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} \quad &= \quad \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \ldots \\ &\quad \ldots + \{f_0, \ldots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\} \end{aligned}$$

- Our construction can be used to quantize these objects.
- Symmetry under Nambu-Poisson maps fixes the general form of a DBI-type effective action for open membranes:

$$S_{\text{bosonic}} = \int \det{}^{\frac{1}{3}}\,[g]\, \det{}^{\frac{1}{6}}\left[g + (B+F)\tilde{g}^{-1}(B+F)^T\right]$$

Jurco, PS (2012)

Conclusion

Summary

- Courant membrane sigma model with R-flux: Target space doubling arises naturally in this model and geometrizes the flux: M is replaced by T^*M, which is interpreted as "phase space" (locally: $\mathbb{R}^n \oplus \mathbb{R}^n$)
- Two string models descent from the membrane model: A closed string model with non-geometric flux (originally studied by the Munich groups) and an open string Poisson sigma model twisted by a geometric 3-form R-flux.
- Perturbative quantization of the twisted Poisson sigma model leads to a nonassociative dynamical star product. Its Jacobiator provides the quantization of the 3-bracket.
- Generalized Seiberg-Witten maps include general coordinate transformations ($\leftrightarrow$ "non-associative gravity), Nambu-Poisson maps and even mappings from associative to non-associative algebras.
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