Remarks on duality symmetries and the space of CFTs

Costas Bachas (Ecole Normale, Paris)

Workshop on "Geometry and Physics"

Munich, November 2012

based on:

CB, I. Brunner, D. Roggenkamp, *arXiv:1205.4647 [hep-th]*
CB, I. Brunner, M. Douglas, L. Rastelli, *in progress*

1. What remains of continuous duality groups?

2. A good (?) distance between CFTs ?

Both subjects have to do with ***metrics on the space of metrics***,
and ***conformal interfaces in 2d***

A famous table : M theory on $\mathbb{R}^{10-d} \times T^{d+1}$

10-d	**U group**	**T group**
9	$SL(2) \times O(1,1)$	$O(1,1)$
8	$SL(3) \times SL(2)$	$O(2,2)$
7	$O(5,5)$	$O(3,3)$
6	$SL(5)$	$O(4,4)$
5	$E_{6(6)}$	$O(5,5)$
4	$E_{7(7)}$	$O(6,6)$

2-derivative supergravity has continuous symmetry

$$G(\mathbb{Z}) \subset G(\mathbb{R})$$

M theory is only invariant under “integer” subgroups

Is there anything in between ?

(Part of) the 2-derivative effective action:

..., Maharana-Schwarz, ...

$$S = M_{\rm Planck}^2 \int d^{10-d}x\sqrt{-g}\left[\frac{1}{8}\mathrm{Tr}(\partial_\mu M^{-1}\partial^\mu M) - \frac{1}{4}(F_{\mu\nu})^T(M^{-1})F^{\mu\nu}\right] \;,$$

where $M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}$ is a $2d \times 2d$ matrix

obeying $M\hat{\eta}M = \hat{\eta}$ with $\hat{\eta} = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$.

This is invariant under the $O(d,d,\mathbb{R})$ transformations:

$$F_{\mu\nu} \mapsto \hat{\Lambda}F_{\mu\nu} \qquad M \mapsto \hat{\Lambda}M\hat{\Lambda}^T$$

with $\hat{\Lambda}^T\hat{\eta}\hat{\Lambda} = \hat{\eta}$.

M parametrizes the homogeneous coset $O(d,d,\mathbb{R})/O(d,\mathbb{R}) \times O(d,\mathbb{R})$

(see C. Hull's talk)

It can be expressed in terms of a *frame matrix*:

$$M = 2\,V^T V \;\leftrightarrow M^{-1} = 2\,(V\hat{\eta})^T\,(V\hat{\eta})\;.$$

which introduces a gauge invariance under $O(d,\mathbb{R}) \times O(d,\mathbb{R})$ transformations of M .

The physical *(canonically-normalized)* gauge fields $F'_{\mu\nu} = V\hat{\eta}F_{\mu\nu}$ do not transform, nor do masses of the corresponding *charged* BHs, consistently with fact that Einstein metric is left unchanged.

The physical charges belong to the Narain lattice $\gamma \in \Gamma^{d,d}$

The integer (winding and momentum) charges are in $\hat{\gamma} \in \mathbb{Z}^d \oplus \mathbb{Z}^d$

To preserve the integer-charge lattice we must require $\hat{\Lambda} \in O(d,d,\mathbb{Z})$

It is a non-trivial fact that the RR-charge lattice, which transforms in the spinor representation, is also left invariant by these transformations.

(see e.g. Obers+Pioline)

This can be shown by worldsheet methods (*see below*).

Consider now transformations $\hat{\Lambda} \in O(d, d, \mathbb{Q})$

These violate charge quantization except when they act on the sublattice $\hat{\Gamma}_{\hat{\Lambda}} = \{\hat{\gamma} : \hat{\Lambda}\hat{\gamma} \in \mathbb{Z}^d \oplus \mathbb{Z}^d\}$

Let $\pi_{\hat{\Lambda}}$ be the projector on this sublattice, and K be the order of the sublattice, $K := |\text{unit cell}(\hat{\Gamma})|$

Example $(d = 1)$

$$O(1, 1, \mathbb{Z}) = \mathbb{Z}_2$$

$$O(1, 1, \mathbb{Q}) = \left\{ \begin{pmatrix} p/q & 0 \\ 0 & q/p \end{pmatrix}, \begin{pmatrix} 0 & q/p \\ p/q & 0 \end{pmatrix} \right\}$$

$$\hat{\Gamma}_{\hat{\Lambda}} = q\mathbb{Z} \oplus p\mathbb{Z} \qquad K = |pq|$$

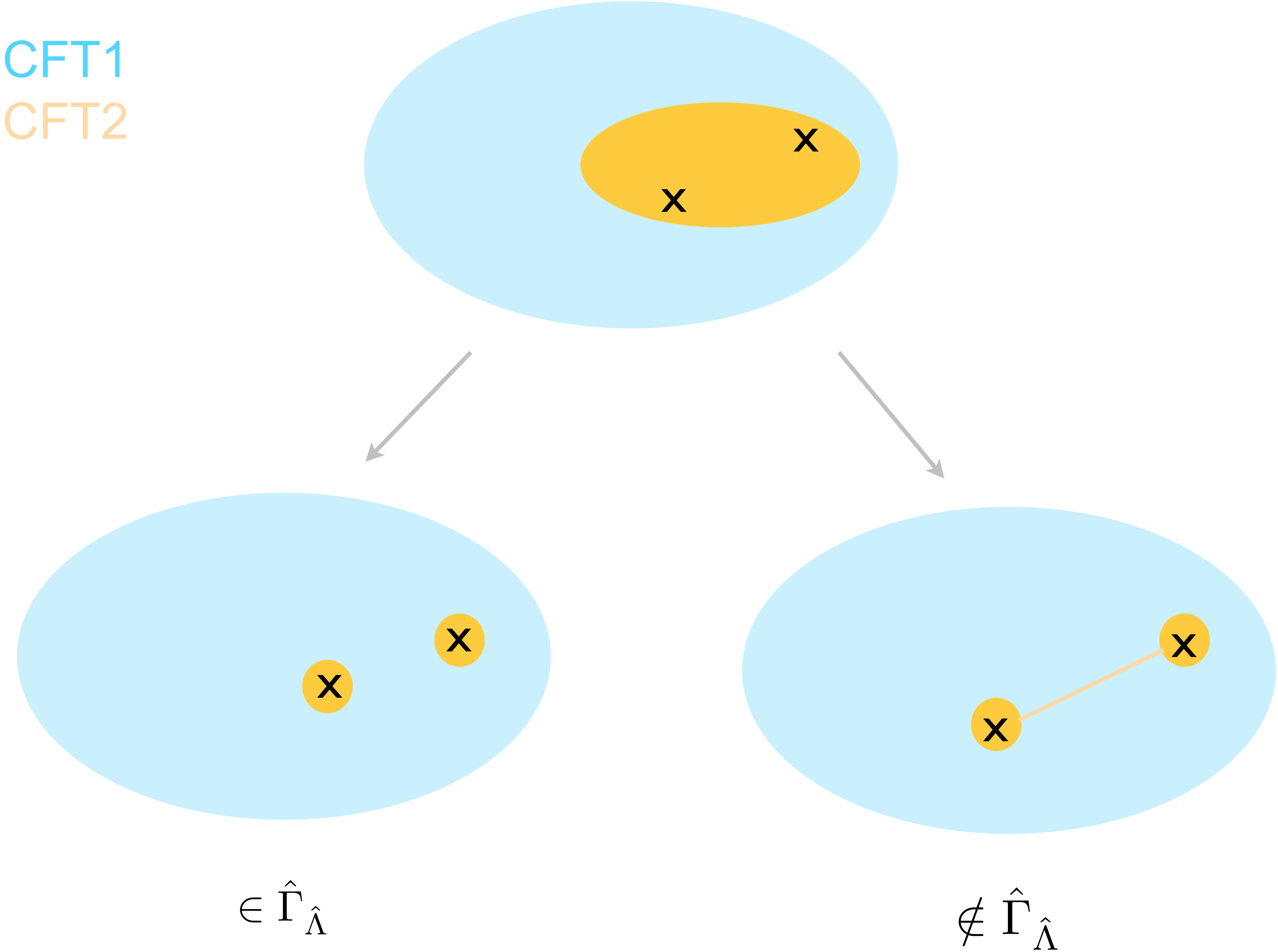
The transformations $\{\hat{\Lambda}\,\pi_{\hat{\Lambda}}\}$ form a semi-group (upon composition) which is a central extension of $O(d,d,\mathbb{Q})$ by the algebra of sublattices.

This is realized on vertex operators by **topological interfaces,**

i.e. interfaces that intertwine both left and right Virasoro algebras:

$$\boxed{\begin{aligned} L_n^{(1)}\, I_{12} &= I_{12}\, L_n^{(2)} \\ \bar{L}_n^{(1)}\, I_{12} &= I_{12}\, \bar{L}_n^{(2)} \end{aligned}}$$

cf also: Petkova, Zuber 00;
CB, Gaberdiel 04;
Fuchs, Gaberdiel, Runkel, Schweigert 07;
CB, Brunner 08


CFT1
CFT2
$\in \hat{\Gamma}_{\hat{\Lambda}}$
$\notin \hat{\Gamma}_{\hat{\Lambda}}$

For $\hat{\gamma} \in \hat{\Gamma}_{\hat{\Lambda}}$ one finds

$$I_{12}\,\mathcal{V}^{(2)}_{\hat{\gamma}} = \sqrt{K}\,\mathcal{V}^{(1)}_{\hat{\Lambda}\hat{\gamma}}$$

*normalization is **fixed** by the generalization of Cardy's condition*

so that the effective string coupling constant transforms as

$$\frac{\lambda_c}{\sqrt{\mathrm{Vol}_d}} =: \lambda_{\mathrm{eff}} \;\mapsto\; \lambda_{\mathrm{eff}}\,\sqrt{|K|}$$

Note the arithmetic nature of this transformation

transform: *moduli* , *integer charges,* *coupling*

d=1 example $R \rightarrow R\,\frac{p}{q}$ $(n,m) \rightarrow (\frac{np}{q}, \frac{mq}{p})$ $\lambda_{\text{eff}} \rightarrow \sqrt{pq}\,\lambda_{\text{eff}}$

invariant: *masses,* *physical charges,* *field equations all orders in* α'

d=1 example $\left|\frac{n}{R} \pm \frac{mR}{\alpha'}\right|^2 + \text{level}$ $(\frac{n}{R}, \frac{mR}{\alpha'})$ $\cdots$

masses, *physical charges,* *field equations all orders in*

This is a special case of a more general story, where topological interfaces exist for any orbifold identification

(Froehlich, Fuchs, Runkel, Schweigert)

In our case the identification is $\boxed{X = X + 2\pi R\,\frac{p}{q}}$

and $\hat{\Gamma}_{\hat{\Lambda}}$ is the lattice of untwisted states.

The semi-group of topological interfaces has also a natural action on D-branes, i.e. boundary states on the world-sheet:

if $$|\mathcal{B}\rangle\!\rangle = \sum_{\alpha=1}^{2^d} n_\alpha |\alpha\rangle\!\rangle$$

elementary D-branes

integer RR charges

$(n_1, \cdots n_{2^d}) := \hat{\gamma}_D$

then $$\hat{\gamma}_D \to \sqrt{K}\, S(\hat{\Lambda})\, \hat{\gamma}_D$$

spinor matrix

puncture $\sim$ hole

(i) The transformation respects **quantization of all RR charges**

i.e. $\sqrt{\mathrm{ind}(\hat{\Lambda})}\, S(\hat{\Lambda}) \in GL(2^d, \mathbb{Z})$

(ii) It reduces to the well-known transformation for $O(d,d,\mathbb{Z})$ transformations, for which $K = 1$

(iii) It leaves **invariant the D-brane mass**, thanks to the transformation of the string coupling

Our favorite d=1 example:

$$|\mathcal{B}\rangle\!\rangle = n_0|\mathrm{D}0\rangle\!\rangle + n_1|\mathrm{D}1\rangle\!\rangle$$

$$\hat{\gamma}_D := \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \qquad \text{or} \qquad \gamma_D := \frac{1}{\lambda_{\text{eff}}} \begin{pmatrix} \sqrt{\frac{\alpha'}{R}} & 0 \\ 0 & \sqrt{R} \end{pmatrix} \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \,,$$

$$\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \to \sqrt{pq} \begin{pmatrix} \sqrt{p/q} & 0 \\ 0 & \sqrt{q/p} \end{pmatrix} \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} = \begin{pmatrix} p\, n_0 \\ q\, n_1 \end{pmatrix}$$

The mass is given by the **physical charge** which is left unchanged.

NB: action on D-brane moduli is here suppressed

Upshot: $O(d,d,\mathbb{Z})$ can be extended to a semi-group extension of $O(d,d,\mathbb{Q})$. This is implemented by topological interfaces, which generalize the (anti)holomorphic contour integrals that implement the CFT automorphisms.

Many open questions:

- the transformation leaves invariant the degeneracy of fundamental string states; is there a natural action on entropy? other observables?

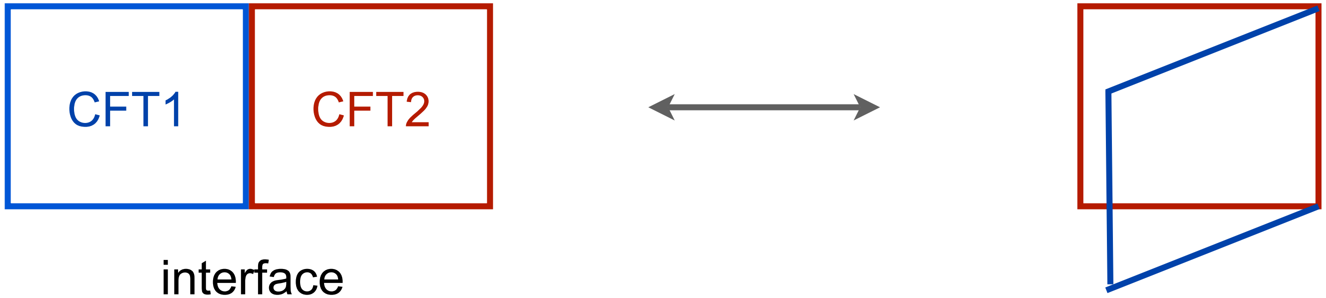
- does this fit into the bigger scheme of U duality ?

NB: $SL(2,\mathbb{Q})$ give planar equivalence, extending mirror symmetry of 3d Chern-Simons theories.

(Assel, CB, Estes, Gomis 12)

To perform these computations we construct the interface operator using the formalism of boundary states:

(Affleck, Oshikawa;
CB, de Boer, Dijkgraaf, Ooguri)



interface

CFT1 x CFT2

boundary

Conformal gluing of currents:

$$\begin{pmatrix} j^1 \\ -\tilde{j}^1 \end{pmatrix} I_{12} = I_{12}\,\Lambda \begin{pmatrix} j^2 \\ -\tilde{j}^2 \end{pmatrix} \qquad \text{for} \qquad \Lambda \in O(d,d,\mathbb{R})$$

(preserves $jj - \tilde{j}\tilde{j}$)

Topological interfaces are those for which $\Lambda \in O(d) \times O(d)$

The integer charges transform by $\hat{\Lambda} = (V_1)^{-1} \Lambda V_2 \in O(d,d,\mathbb{Q})$

The operator in the bosonic theory reads $I_{12} = \prod_{n \geq 0} I_{12}^{n,\mathrm{bos}}$

where: $I_{12}^{n,\mathrm{bos}} = \exp\Big(\frac{1}{n} (j^1_{-n} \mathcal{O}_{11} \tilde{j}^1_{-n} - j^1_{-n} \mathcal{O}_{12} j^2_n - \tilde{j}^1_{-n} \mathcal{O}^t_{21} \tilde{j}^2_n + a^2_n \mathcal{O}^t_{22} \tilde{j}^2_n) \Big)$

$$\Lambda := \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \qquad \mathcal{O}(\Lambda) = \begin{pmatrix} \Lambda_{12}\Lambda_{22}^{-1} & \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21} \\ \Lambda_{22}^{-1} & -\Lambda_{22}^{-1}\Lambda_{21} \end{pmatrix} .$$

$$I_{12}^{0,\mathrm{bos}} = \sqrt{\mathrm{ind}(\hat{\Lambda})\, |\Lambda_{22}|} \sum_{\hat{\gamma} \in \mathbb{Z}^{d,d}} e^{2\pi i \varphi(\hat{\gamma})} |\hat{\Lambda}\hat{\gamma}\rangle \langle \hat{\gamma}| \, \Pi_{\hat{\Lambda}}$$

***g*-factor**

NB: there are analogous expressions for the type-II superstring.

- Topological interfaces **minimize** the g-factor, if one keeps the discrete data $\hat{\Lambda}$ and V_1 fixed, and varies the other modulus V_2 :

$$\boxed{g \geq g_{\rm top} = \sqrt{{\rm ind}(\hat{\Lambda})}}$$

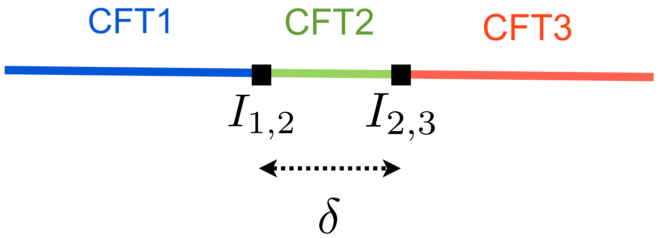
Space-time supersymmetry implies the topological property

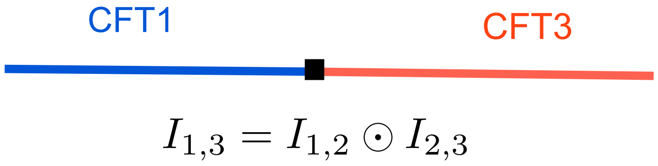
- The T-duality interfaces are topological with $g = 1$

- Deformed-identity interfaces, which can be used for transport on moduli space, have $\hat{\Lambda} = \mathbb{I}$ and $g = \sqrt{|\Lambda_{22}|}$

where $\Lambda = V_1^{-1} V_2$ is the boost of the Narain lattice.

Interfaces can be added and **fused** :





$I_{1,3} = I_{1,2} \odot I_{2,3}$

$$I_{12} \odot I_{23} \ := \ \lim_{\delta \to 0} \mathcal{R}_\delta [I_{12}\, e^{-\delta H}\, I_{23}]$$

This can be computed for free fields, but in general difficult.

(stat-mech applications, eg. 2D Ising model)

Distance between CFTs

When are two QFTs close to each other ?

Friedan 85; ...... ;

Kontsevitch, Soibelman arXiv:math.SG/0011041 ; http://www.math.ksu.edu/ soibel/nc-riem-3.pdf

Douglas [hep-th 1005.2779]

The question is interesting, and not intuitively clear:

- Are "large-volume" limits the **only large-distance limits**?

gap on spectrum of operator dimensions, Δ ,vanishing

- Are all finite-distance limits QFTs? (conifold? limit of minimal models? ...)

- Is there a finite # of CFTs with fixed c, and gap ("precompactness")

Even in the simpler case of Riemannian manifolds, vast subject

Gromov, Cheeger .....

On space of diffeomorphic manifolds, we know from quantum gravity:

$$||\delta g||_U^2 \equiv \mathcal{N} \int_M \sqrt{g}\, \left(g^{ik}g^{jl} + \alpha g^{ij}g^{kl}\right)\, \delta g_{ij}\delta g_{kl}$$

This is positive-definite for $\mathcal{N} > 0,\ \alpha > -1/d,$ but *not reparametrization-invariant*

(non-zero for $\delta g_{ij} = \nabla_{(i}\xi_{j)}$)

Can cure this by minimizing over ξ , i.e. choose δg in harmonic gauge.

The normalization is not a priori fixed. The one appearing in Kaluza-Klein theory is:

$$||\delta g||_N^2 \equiv \frac{\int_M \sqrt{g}\, \left(g^{ik}g^{jl} + \alpha g^{ij}g^{kl}\right)\, \delta g_{ij}\delta g_{kl}}{\int_M \sqrt{g}}$$

These (path) distances have drawbacks:

- non-diffeomorphic manifolds (*e.g. different topology*)?
- hard to manipulate, so as to answer previous questions

Mathematicians have devised alternative distances, e.g.

- approximate manifolds by **union of balls**, ignoring structure below some minimal scale
- **embedding metrics**, e.g. $d_{\text{Hausdorff}}(X,Y) = \max_{x\in X} \min_{y\in Y} d(x,y)$

minimized over all isometric embeddings $X, Y \subset \mathbb{R}^N$

Gromov-Hausdorff distance

(non-constructive, but good properties)

Can we define analogous metrics for QFTs? or at least 2D CFTs?

We need to satisfy (at least) the axioms:

$$\begin{aligned} d(T_1,T_2) &\geq 0 && \text{(with equality only if } T_1 \cong T_2\text{)}, \\ d(T_1,T_2) &= d(T_2,T_1), && \\ d(T_1,T_2)+d(T_2,T_3) &\geq d(T_1,T_3) && \text{(the triangle inequality).} \end{aligned}$$

On connected moduli spaces we have the Zamolodchikov metric(s):

$$\|\delta T\|^2_{UZ} = \mathcal{Z}(T)\,|x|^4 \langle \mathcal{O}_{\delta T}(0)\mathcal{O}_{\delta T}(x)\rangle$$

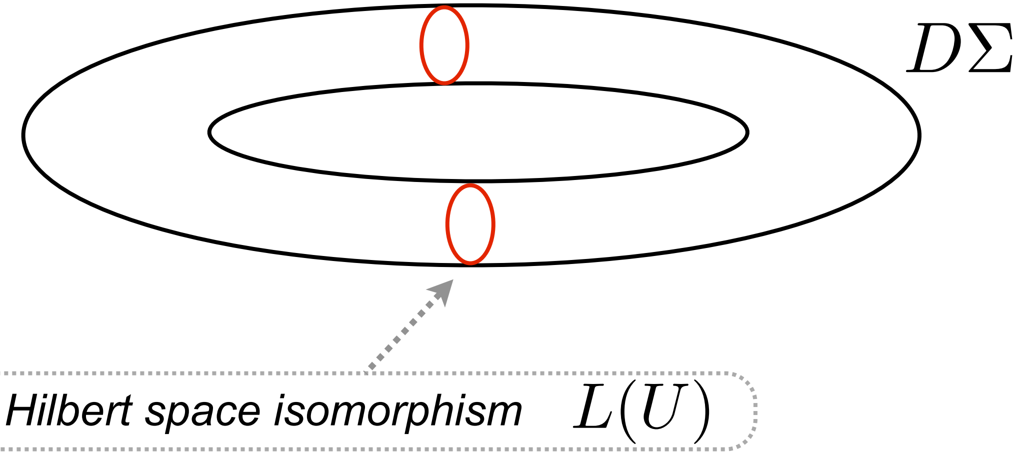
$$\|\delta T\|^2_{Z} = \#\,|x|^4 \langle \mathcal{O}_{\delta T}(0)\mathcal{O}_{\delta T}(x)\rangle$$

These reduce to previous diffeo-invariant metrics in the geometric large-volume limit. They can be extended to “small” RG flows.

But (even more) restrictive than for manifolds: most CFTs are at infinite separation in the Z-distance.

Douglas proposed various embedding metrics, e.g.

$$d_\Sigma(T_1,T_2)^2 = \min_U \left[\mathcal{Z}_{D\Sigma}(T_1) + \mathcal{Z}_{D\Sigma}(T_2) - 2\mathcal{Z}_{D\Sigma}(T_1|L(U)|T_2)\right]$$



Hilbert space isomorphism $L(U)$

example: using the canonical isomorphism of c=1 circle theories

$$d_{\mathrm{cyl}}(R_1,R_2)^2 \le \frac{1}{\eta(q)^2}\sum_{m,n}\left(q^{n^2/R_1^2+4m^2R_1^2} - q^{n^2/R_2^2+4m^2R_2^2}\right)^2$$

Problems: -- unphysical (non-local gluing), hard to calculate

-- first axiom not obeyed, e.g. for $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$

$$(\theta_2^8 + \theta_3^8 + \theta_4^8)^2/4\eta^{16} = (\theta_2^{16} + \theta_3^{16} + \theta_4^{16})/2\eta^{16}$$

Can fix it with higher-genus surface, but make first problem even worse

-- does not reduce to Zamolodchikov metric for nearby points

New proposal: the "g-distance"

$$d_g(T_1, T_2) = \left(\min_{\mathcal{I} \in S} \ \log g(\mathcal{I}(T_1, T_2)) \right)^{1/2}$$

?

Comes close, but does not really work !

But: learn interesting properties of the g-function;

Can it be fixed?

◆ Only the <u>second axiom</u> is automatic. ✓

◆ The first axiom requires an appropriate choice of *S*

We conjecture (but cannot prove) that :

weak $$S_I = \{\mathcal{I}(T_1, T_2) \mid \mathcal{I} \odot \mathcal{I}^\dagger \text{ and } \mathcal{I}^\dagger \odot \mathcal{I} \text{ topological}\}$$

or at least

remove totally-reflecting interfaces

strong $$S'_I = \{\mathcal{I}(T_1, T_2) \mid \mathcal{I} \odot \mathcal{I}^\dagger = 1_{T_1} \text{ and } \mathcal{I}^\dagger \odot \mathcal{I} = 1_{T_2} \}$$

suffice to imply $g \geq 1$, and $g = 1$ only for

CFT automorphisms.

◆ Third axiom fails, but can (probably) be fixed.

Two properties

- For continuous moduli spaces, and infinitesimal deformations, the *g*-distance reduces to the Zamolodchikov metric :

$$\log g(I(T_t, T_{t+\Delta t})) = \frac{\pi^2}{4}\, g^{(Z)}_{ij}(t)\Delta t^i \Delta t^j + O(\Delta t^3)$$

Proof: $$g = N_\epsilon < e^{\Delta t^j \int_0^{2\pi} d\sigma \int_0^\infty d\tau\, O_j} >$$

subtract divergent self-energy

$$\log g = \log N_\epsilon + \frac{1}{2}\Delta t^j \Delta t^k g^{(Z)}_{jk} \int_{\tau>0}\int_{\tau'>0} d^2z d^2z' \left|\frac{e^z e^{z'}}{(e^z - e^{z'})^2}\right|^2 + O(\Delta t^3)$$

$$\int_{\tau>0} d^2z \int_{\tau'>0} d^2z' \frac{\partial}{\partial \bar{z}'}\left[\frac{e^{\bar{z}}}{(e^{\bar{z}} - e^{\bar{z}'})}\frac{e^z e^{z'}}{(e^z - e^{z'})^2}\right] = -\pi \int_{\tau>0} d^2z \left[\frac{e^{\bar{z}}}{(e^{\bar{z}} - 1)}\frac{e^z}{(e^z - 1)^2}\right]$$

$$\pi \int_{\tau>0} d^2z \frac{\partial}{\partial z}\left[\frac{e^{\bar{z}}}{(1 - e^{\bar{z}})}\frac{1}{(e^z - 1)}\right] = -\frac{\pi}{2}\int_0^{2\pi} d\sigma \left[\frac{e^{\bar{z}}}{(1 - e^{\bar{z}})}\frac{1}{(e^z - 1)}\right]_{\tau=\epsilon}$$

$$-\pi^2 \sum_{n=1}^{\infty} e^{-2n\epsilon} = -\cancel{\frac{\pi^2}{2\epsilon}} + \frac{\pi^2}{2} + O(\epsilon)$$

QED

maps UV to IR operators

Also checked this for the RG-flow interface of Gaiotto 12

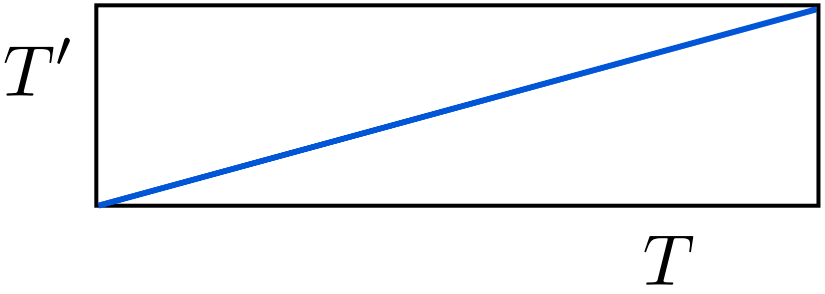
Not clear for N=2 flows, but these are “long” (Brunner, Roggenkamp 07)

◆ On the Kahler- or complex-structure moduli spaces of CY manifolds, the *g*-distance reduces to (is bounded by) Calabi’s diastasis function

Consider e.g. the 2-torus CFTs with moduli

$$G = \frac{\tau_2}{\rho_2}\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & |\rho|^2 \end{pmatrix} \quad , \qquad B = \begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix}$$

The deformed-identity interface is (after folding) a diagonal D2-brane:



with g-factor (“mass in Einstein-frame”) :

$$g = \frac{1}{\sqrt{4\tau_2\tau_2'}}\,\mathrm{det}^{1/2}(G + G' + B - B')$$

$$\Longrightarrow \log g = \log\left[\frac{(\tau-\bar{\tau}')(\tau'-\bar{\tau})}{(\tau-\bar{\tau})(\tau'-\bar{\tau}')} + \frac{(\rho-\bar{\rho}')(\rho'-\bar{\rho})}{(\rho-\bar{\rho})(\rho'-\bar{\rho}')} - 1\right] \ .$$

If $\rho = \rho'$ (only complex-structure deformation) :

$$d^2(\tau,\tau') = \log g = -\log(\tau-\bar{\tau}) - \log(\tau'-\bar{\tau}') + \log(\tau-\bar{\tau}') + \log(\tau'-\bar{\tau})$$

i.e.

$$d^2(t_1,t_2) = K(t_1,\bar{t}_1) + K(t_2,\bar{t}_2) - K(t_1,\bar{t}_2) - K(t_1,\bar{t}_2)$$

Calabi’s diastasis:

- can be defined for any Kahler manifold
- it is a function (Kahler-Weyl independent)
- preserved under restriction to submanifolds

Story extends to general CY moduli spaces.

The deformed identities are special Lagrangian, or holomorphic submanifolds of $M_1 \times M_2$, which is also CY but “more special”

holonomy $SU(n) \times SU(n) \subset SU(2n)$

If one deforms the complex structure, the supersymmetric brane is s.L. with the pullback of $\omega_1 - \omega_2$ vanishing, and the **calibrating *2n*-form** $\Omega_1 \wedge \bar{\Omega}_2$ constant . The normalized brane volume then reads

$$d^2(t_1, t_2) = -K(t_1, \bar{t}_1) - K(t_2, \bar{t}_2) + 2 \log | \int_M \Omega_1 \wedge \bar{\Omega}_2 |$$

Einstein-frame normalization

volume

which, using $K(t, \bar{t}) = -\log \int_M \Omega(t) \wedge \bar{\Omega}(\bar{t})$, is Calabi’s diastasis.

In Kahler normal coordinates $K(t,\bar{t}) = \sum_{\alpha} |t^\alpha|^2 - \frac{1}{4} R_{\alpha\bar{\beta}\gamma\bar{\delta}} t^\alpha \bar{t}^{\bar{\beta}} t^\gamma \bar{t}^{\bar{\delta}} + \ldots$

so $d^2(t_1,t_2) = |t_1 - t_2|^2 - \frac{1}{4} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (t_1^\alpha t_1^\gamma - t_2^\alpha t_2^\gamma)(\bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}}) + \ldots$

and $d(t_1,t_2) = |t_1 - t_2| - \frac{1}{8|t_1 - t_2|} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (t_1^\alpha t_1^\gamma - t_2^\alpha t_2^\gamma)(\bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}}) + \ldots$

The triangle inequality to this order (with $t_3 = 0$) reads:

$$|t_1| + |t_2| -| t_1 - t_2| \geq -\frac{1}{8} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \left(\frac{1}{|t_1 - t_2|} (t_1^\alpha t_1^\gamma - t_2^\alpha t_2^\gamma)(\bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}}) - \frac{1}{|t_1|} t_1^\alpha t_1^\gamma \bar{t}_1^{\bar{\beta}} \bar{t}_1^{\bar{\delta}} - \frac{1}{|t_2|} t_2^\alpha t_2^\gamma \bar{t}_2^{\bar{\beta}} \bar{t}_2^{\bar{\delta}} \right) .$$

Taking $t_1 = -t_2 = x$ then gives $R_{x\bar{x}x\bar{x}} < 0$

everywhere negative sectional curvature

This condition is unfortunately violated, in particular near the conifold point

Candelas, De La Ossa, Green, Parkes

so the triangle inequality does not hold.

Fixing the triangle inequality

If $d(x,y)$ satisfies first two axioms, then define:

$$\boxed{d_{t,2}(x,y) = \min_{z}\left\{d(x,z) + d(z,y)\right\}}$$

If $d(x,y)$ satisfies the inequality, then $d_{t,2}(x,y) = d(x,y)$

else, first step of “improvement”. Repeat procedure and finally

$$d_{t,n}(x,y) = \min_{z_1,\ldots,z_n}\left\{d(x,z_1) + d(z_1,z_2) + \ldots + d(z_{n-1},z_n) + d(z_n,y)\right\}.$$

$$d_t(x,y) = \liminf_{n\longrightarrow\infty}\ d_{t,n}(x,y).$$

NB: No guarantee of convergence, and if it does limit may have bad

properties (e.g. $d_{t,n}(x,y) = 0 \quad \text{for} \quad x \neq y$)

But in many cases, this restores the triangle inequality.

Summary

1. There exists a natural worldsheet extension of T-duality symmetry, generated by topological interfaces on the worldsheet.

2. A simple proposal for a distance between CFTs, via the *g*-factor of conformal interfaces, comes close but requires a fix.