Well-Posed Boundary Data + Problems for Einstein Field Equations

M. Anderson

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Einstein Metrics

$$(M, g)$$

$$E(g) = Ric_g - \frac{R}{2} g + \Lambda g = 0$$

Riemannian : Elliptic

Parabolic $\longleftrightarrow$ Ricci flow

Lorentzian : Hyperbolic

Riemannian Metrics

M^{n+1}, $\partial M \neq \emptyset$: boundary value problems

In $\mathring{M}$ - interior

$E(g) = 0$ ~ elliptic equation

- gauge issue : divergence-free gauge

$\tilde{g}$: background metric

$$\Phi_{\tilde{g}}(g) = E(g) + \delta^* \delta_{\tilde{g}}(g) \quad : \text{elliptic operator}$$

Boundary condition : $\delta_{\tilde{g}}(g) = 0$ on ∂M

Then

$$\Phi_{\tilde{g}}(g) = 0 \text{ on } M \Rightarrow E(g) = 0 \text{ on } M$$

$\mathcal{E}$: moduli space of Einstein metrics on M

$= \mathcal{E}^{m,\alpha}(\Lambda, M)$ "space of solutions mod gauge equivalence"

$= E^{m,\alpha} / \mathrm{Diff}_1^{m+1,\alpha}$

<u>Thm</u> $\mathcal{E}$ is a smooth Banach manifold

- have good "local" behavior
  - linearization stability

- Parametrize $\mathcal{E}$ by "boundary data"

  - geometric (γ, K)

γ: induced metric on ∂M

K: 2nd fund. form, extrinsic curvature

Boundary Data

Dirichlet Data $\gamma = g|_{\partial M}$

$$\Pi_D : \mathcal{E} \longrightarrow Met(\partial M)$$
$$[g] \longrightarrow \gamma$$

C^∞ smooth

existence/uniqueness of solutions to Dirichlet BVP $\longleftrightarrow$ surjectivity/injectivity of Π_D

Not elliptic BVP

Π_D is not Fredholm

∞-dim. cokernel

Not well-posed BVP

- strong contrast with other geometric/physical field equations

Hamiltonian Constraint (Diffeo invariance)

$$|K|^2 - (trK)^2 + R_\gamma = 2\Lambda$$

$\gamma \in C^{m,\alpha} \Rightarrow R_\gamma \in C^{m-1,\alpha} \Rightarrow$ generic γ not realized in $Im\,\Pi_D$

<u>Contrast</u> AH (AdS) case:

Dirichlet BVP at <u>conformal</u> <u>infinity</u>

is elliptic/Fredholm   Graham-Lee

Biquard

Bringing boundary in to "finite distance"

$\sim$ RG flow ("radial variable")

$\sim$ Fluid/Gravity correspondence

- Bratten-Camps-Loganayagam-Rangamani   1106.2577
- Marolf-Rangamani   1201.1233
- Bredberg-Keeler-Lysov-Strominger   1101.2451
- Wiseman, Headrick, Figueras, ...

<u>Note</u>: Simplest Case of Dirichlet BVP

$n=2$, $\Lambda=0$

g = flat metric on $M^3 = B^3$

Dirichlet problem $\longleftrightarrow$ isometric immersion problem

$(S^2, \gamma) \longrightarrow (\mathbb{R}^3, \delta)$

notoriously difficult

Elliptic Boundary Data

$[\gamma], H$ (mixed Dirichlet-Neumann data)

conformal class of γ — mean curvature

$$\Pi : \mathcal{E}^{m,\alpha} \longrightarrow \mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$$

$$[g] \longrightarrow ([\gamma], H)$$

Fredholm, index 0

"well-posed"

(or $[\gamma]_B$, $tr_\sigma A$)

$\Pi_B > 0 \quad \sigma > 0$

"$B - tr_\gamma B \cdot \gamma$: conjugate momentum

- Static Vacuum Equations

$$u Ric = D^2 u$$
$$\Delta u = 0$$

$\Pi(g, u) = (\gamma, H)$ - Bartnik boundary data

- elliptic, Fredholm index 0

Global Non-linear Problems

Π surjective ?

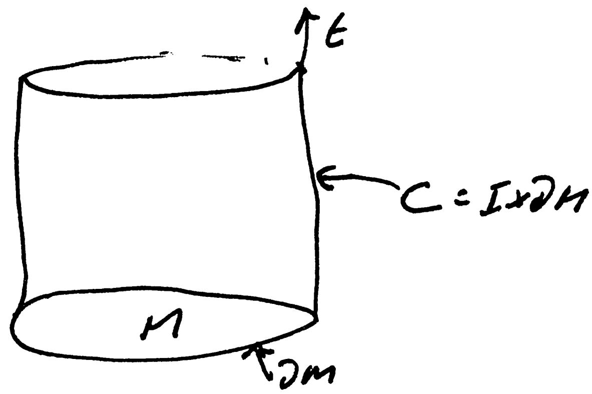
injective ?

Parabolic Equation

Ricci flow: $\frac{dg}{\partial t} = -2\,\mathrm{Ric}_g$

on M^{n+1}, $\partial M \neq \emptyset$

Initial Boundary Value Problem



Initial data = initial metric
$g_0 = g|_{t=0}$

Boundary Data ??

Sporadic Earlier work

Y. Shen '96

S. Brendle '02

A. Pulemotov '10, J. Cortissoz '09

Dirichlet boundary data: not well posed.

Thm (P. Gianniotis)

The IBVP for Ricci flow is well-posed (existence + uniqueness of solutions for some time $T > 0$) for boundary data

$$([\gamma], H) \text{ on } C = [0,T] \times \partial M$$

– assuming corner = compatibility conditions hold.

Continuation criterion: $|Rm|_{L^\infty(M)} + |K|_{L^\infty(\partial M)} \leq C \Rightarrow$ solution continues (smooth ∂ data)

Lorentz Metrics / Hyperbolic Eqns

$$Ric_g - \frac{R}{2} g + \Lambda g = 0$$

Cauchy problem = initial value problem

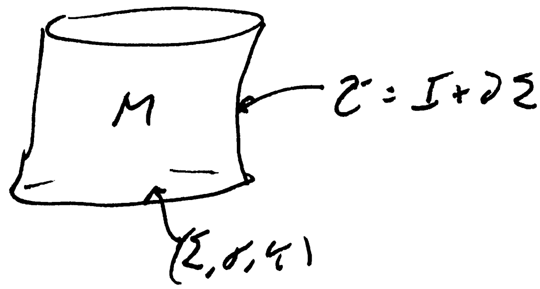
$$(\Sigma, \gamma, \kappa)$$

initial slice 1st & 2nd fund. forms

well-posed (Choquet-Bruhat)

Initial Boundary Value Problem

$$M = I \times \Sigma$$



Find geometric boundary data on $\mathcal{C} = I \times \partial\Sigma$ - involving $(\gamma_{\mathcal{C}}, \kappa_{\mathcal{C}})$

s.t. I.B.V.P. is well-posed.

(H. Friedrich '09)

H. Friedrich - G. Nagy ('99)
H-O Kreiss, J. Winicour ('06)
O. Reula, ...

} well-posed IBVP for gauge-dependent boundary data

boundary data isometric $\overset{?}{\Rightarrow}$ solutions isometric

Dirichlet ∂ data on $\mathcal{T}$: $\gamma_{\mathcal{T}}$ fixed : not well-posed

Mixed Dirichlet-Neumann data (elliptic case) : $([\gamma_{\mathcal{T}}], H)$: unknown - probably No

One motivation for solution of geometric IBVP:

"fully adequate" definition of quasi-local Hamiltonian (energy, mass...) depends on solution of IBVP

Brown-York Hamiltonian

(Hawking-Horowitz, Kijowski, Epps, Liu-Yau, Wang-Yau, ...)

E-H action + G-H-Y boundary term

$$I(g) = \int_M R_g \, dV_g + 2\int_{\mathcal{C}} \kappa \, dV_\gamma \qquad \kappa = tr_\gamma K_{\mathcal{C}}$$

$$\delta I(h) = -\int_M \langle E(g), h \rangle - \int_{\mathcal{C}} \langle \pi_{\mathcal{C}}, h^T \rangle dV_\gamma$$

"conjugate momentum" $= K_{\mathcal{C}} - \kappa\gamma$

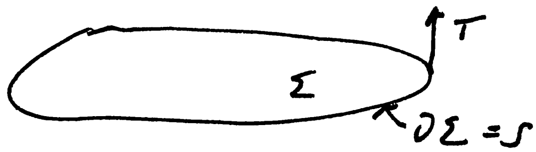
h^T : variation of $\gamma_{\mathcal{C}}$ on $\mathcal{C}$
= Dirichlet boundary data

Have well-defined variational formulation:

On-shell moduli space = Einstein metrics with fixed boundary metric γ & action is functionally differentiable on $Met_\gamma(M)$

Hamilton-Jacobi analysis

On-shell Hamiltonian = variation of on-shell action in direction of time-like unit normal : $h^T = T \cdot T$



$$\Pi_{\tau}(T, T) = H_S .$$

$$\mathcal{H}_{BY} = - \int_S H_S \, dA$$ "bare" BY Hamiltonian (on-shell)

Problem Hamiltonian vector field

$$V_{\mathcal{H}} = \left(\frac{\delta H}{\delta \pi}, - \frac{\delta H}{\delta g} \right)$$

not integrable : equations of motion (Einstein eqns)

$$\frac{\delta H}{\delta \pi} = \dot{g} , \quad \frac{\delta H}{\delta g} = - \dot{\pi}$$

not solvable (well-posed) with these boundary conditions

Q : Can one find ∂-term for E-H action s.t. corresponding IBVP is well-posed.

If so, can read off resulting Hamiltonian