# Superconducting superstrings

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Based on work with S. Gubser, C. Herzog, F. Rocha, and T. Tesileanu

`arXiv:0907.3510`, `arXiv:0908.0011`

*Munich, September 7, 2009*

# 1. Outline

- A few facts about ”superconducting” black holes and the Abelian Higgs model in AdS.

- Uplift of the Abelian Higgs model to type IIB superstring theory.

- (Some) Instabilities of $\mathcal{N} = 4$ SYM at finite temperature and R-charge chemical potential.

# 2. Superconducting black holes

One of the simplest setups where we see "superconducting" black holes is the Abelian Higgs model in anti-de Sitter (AdS) space: [Gubser '08; Hartnoll, Herzog, Horowitz '08

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} - \frac{1}{4} F_{\mu\nu}^2 - |\partial_\mu \psi - iqA_\mu \psi|^2 - V(|\psi|) \right] . \quad (1)$$

Its dynamical fields are the metric $g_{\mu\nu}$, the gauge field $A_\mu$, and the complex scalar field $\psi$ with charge $q$ and potential $V(|\psi|)$.

- Extrema of (1) include Reissner-Nordström AdS (RNAdS) black holes with $\psi = 0$. Radius of AdS is $L$.
- There may also be solutions with $\psi \neq 0$ (but nevertheless normalizable), which break *spontaneously* the gauge symmetry by choosing a phase for $\psi$.
- The solutions that break the $U(1)$ gauge symmetry are referred to as "superconducting" by analogy with the regular superconductors.

## Why is this interesting?

1. It shows that the "no hair" theorem from flat space doesn't work in AdS.

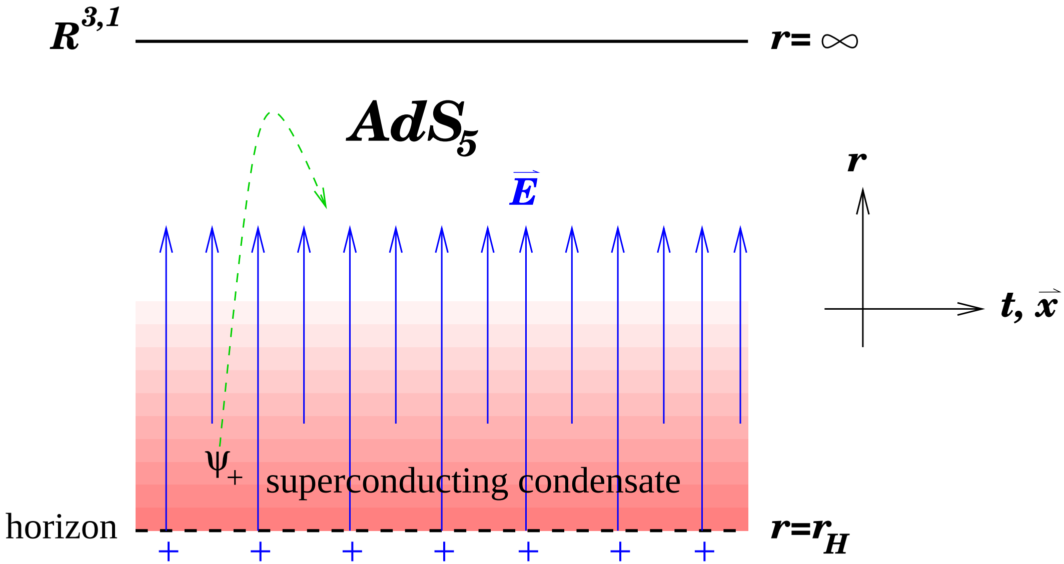


Figure 1: *Superconducting condensate floating above an AdS black hole.*

2. Using AdS/CFT, we can interpret the superconducting black holes as superconducting (more correctly superfluid) states in a strongly coupled dual field theory in $3+1$ dimensions.

One can look for extrema of (1) of the form

$$ds^2 = e^{2a(z)}\left[-h(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{h(z)}\right] \qquad A = \Phi(z)dt$$
$$\psi = \psi(z) \quad \text{(real)}\,. \tag{2}$$

- Black hole horizon is at $z = z_H$ where $h(z_H) = 0$. The space outside the black hole is at $0 < z < z_H$.
- RNAdS corresponds to $\psi = 0$ as well as

$$e^a = \frac{L}{z} \qquad h = 1 - \left(1 + \frac{\mu^2 z_H^2}{3}\right)\frac{z^4}{z_H^4} + \frac{\mu^2}{3}\frac{z^6}{z_H^4} \qquad \Phi = \mu L\left(1 - \frac{z^2}{z_H^2}\right)\,. \tag{3}$$

- One can expect symmetry breaking solns to exist if $\psi$ has zero-modes in RNAdS.
- There are no known analytical solutions with symmetry breaking.
- The Abelian Higgs lagrangian (1) is an effective 5d lagrangian. Can it be obtained from string theory? How?

# 3. A universal consistent truncation of IIB SUGRA

- Consistent truncations come up all the time, e.g. restrict planetary motion to a plane.
- To construct a consistent truncation, one needs to express the 10d fields of type IIB SUGRA in terms of the 5d fields (metric, gauge field, complex scalar). This ansatz solves the 10d SUGRA eoms iff the 5d fields satisfy some effective 5d equations that can be derived from an Abelian Higgs-like lagrangian.
- Why embed the Abelian Higgs model into string theory (or type IIB SUGRA)? Because this way we'd know which CFT we're looking at.
- The CFTs I'll focus on are dual to $AdS_5$ times a $5$-dimensional Sasaki-Einstein manifold $SE_5$. (For $AdS_4$ embeddings see [Gauntlett, Kim, Verela, Waldram '09; Denef, Hartnoll '09; Gauntlett, Sonner, Wiseman '09].)
  - These are $\mathcal{N} = 1$ quiver gauge theories with non-zero superpotentials $\mathcal{W}$.

| $10$d type IIB SUGRA<br>$U(1)_R$ realized geometrically | $\Longrightarrow$ | $5$d AdS<br>*gauged* $U(1)$ | $\Longrightarrow$ | $4$d $\mathcal{N} = 1$ CFT<br>*global* $U(1)_R$ |
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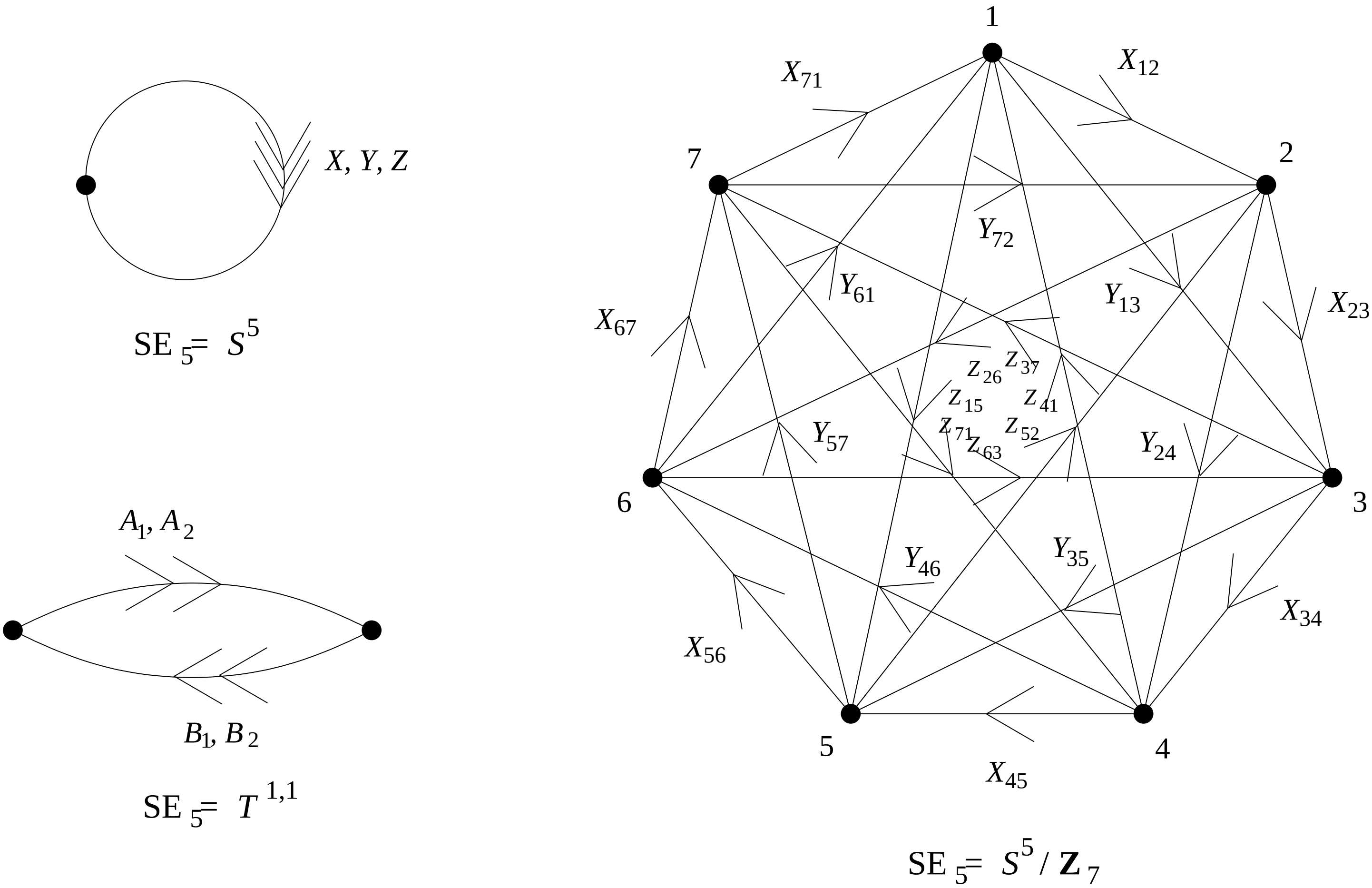



Figure 2: *Some quiver gauge theories dual to* $AdS_5 \times SE_5$.

The *bosonic* field content of type IIB SUGRA is:

- NS-NS fields: $G_{MN}$, $\Phi$, $B_2$ (w/ field strength $H_3 = dB_2$).
- R-R fields: $C$, $C_2$, $C_4$ (w/ field strengths $F_1 = dC$, $F_3 = dC_2$, and $F_5 = dC_4$).

Type IIB SUGRA action is quite complicated (see, e.g. [Polchinski '98]):

$$\begin{aligned} S_{\mathrm{IIB}} = {} & \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & - \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) - \frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3\,, \end{aligned} \tag{4}$$

where $\tilde{F}_3 \equiv F_3 - CH_3$ and $\tilde{F}_5 \equiv F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$.

- Also need to impose $*\tilde{F}_5 = \tilde{F}_5$ *after* deriving the eoms from the action.

Write $\boxed{\begin{array}{c}\text{10d fields}\\ G_{MN},\ \Phi,\ B_2\\ C,\ C_2,\ C_4\end{array}}$ in terms of $\boxed{\begin{array}{c}\text{5d fields}\\ g_{\mu\nu}\\ A_\mu\\ \eta \text{ and } \theta \text{ (modulus and phase of } \psi)\end{array}}$.

Embedding given in [Gubser, Herzog, SSP, Tesileanu '09] (see also [Corrado, Pilch, Warner '02; Pilch, Warner '01 '02]).

Start with the easy part: $\Phi = C = 0$.

First write $SE_5$ as a $U(1)$-fibration over a Kähler-Einstein two-fold:

$$ds^2_{SE_5} = ds^2_{KE_2} + \zeta^2 \qquad \zeta \equiv d\psi + \sigma\,. \tag{5}$$

Then $G_{MN}$ is

$$ds^2 = \cosh\frac{\eta}{2}\, ds_5^2 + \frac{L^2}{\cosh\frac{\eta}{2}}\left[ds^2_{KE_2} + \cosh^2\frac{\eta}{2}(\zeta^A)^2\right]\,, \tag{6}$$

where

$$\zeta^A \equiv \zeta + \frac{1}{L}A\,. \tag{7}$$

For $SE_5 = S^5$, the KE two-fold is $\mathbf{CP}^2$ and (5) is the Hopf fibration.

- Eq. (6) has the $U(1)$ fiber stretched.

- Cone over $SE_5$ is Calabi-Yau $CY_3$: $ds^2_{CY_3} = dr^2 + r^2 ds^2_{SE_5}$.
- Consider holomorphic $3$-form $\hat{\Omega}_3$ normalized so that $\hat{\Omega}_3 \wedge \bar{\hat{\Omega}}_3 = 8\mathrm{vol}_{CY_3}$. For $SE_5 = S^5$, we have $CY_3 = \mathbf{C}^3$ and $\hat{\Omega}_3 = dz^1 \wedge dz^2 \wedge dz^3$.
- This form decomposes as

$$\hat{\Omega}_3 = r^2 dr \wedge \Omega_2 + (3\text{-form on SE base})\,. \tag{8}$$

$B_2$ and $C_2$ are given by

$$B_2 + iC_2 = L^2 \tanh\frac{\eta}{2} e^{i\theta}\Omega_2\,. \tag{9}$$

Lastly, $\tilde{F}_5$ is

$$\tilde{F}_5 = (1 + *_{10})\left[\frac{L^4(\cosh\eta - 5)}{2\cosh^2\frac{\eta}{2}}\zeta^A \wedge \omega^2 + L^3 F \wedge \zeta^A \wedge \omega - \frac{L^2}{2}(*_5 J) \wedge \zeta^A\right], \tag{10}$$

where $F \equiv dA$ and $J \equiv \frac{1}{L}\sinh^2\eta\left(d\theta - \frac{3}{L}A\right)$.

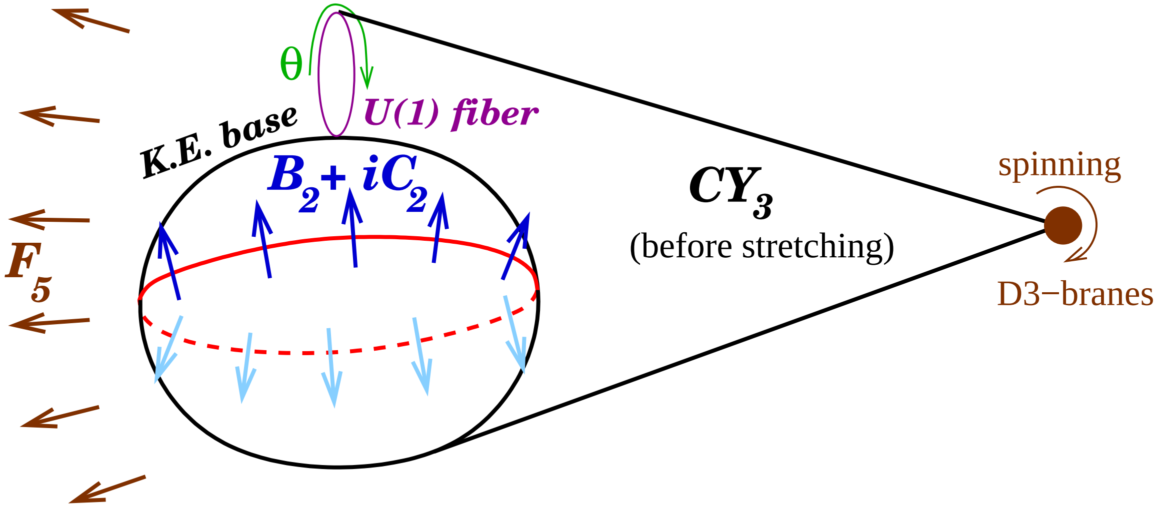



Figure 3: *A schematic view of the 10d geometry.*

By plugging this ansatz into the type IIB eoms, we obtain equations for $g_{\mu\nu}$, $A_\mu$, $\eta$, and $\theta$ that can also be derived from the $5$d *effective* lagrangian

$$\begin{aligned}\mathcal{L}_{5\mathrm{d}} &= R - \frac{3}{4}F_{\mu\nu}^2 - \frac{1}{2}\left[(\partial_\mu\eta)^2 + \sinh^2\eta\left(\partial_\mu\theta - \frac{3}{L}A_\mu\right)^2\right] \\ &\quad + \frac{3}{L^2}\cosh^2\frac{\eta}{2}\left(5 - \cosh\eta\right) + \textbf{(Chern-Simons)}\,.\end{aligned} \tag{11}$$

- Can construct hairy black hole solutions with non-zero $\eta$.

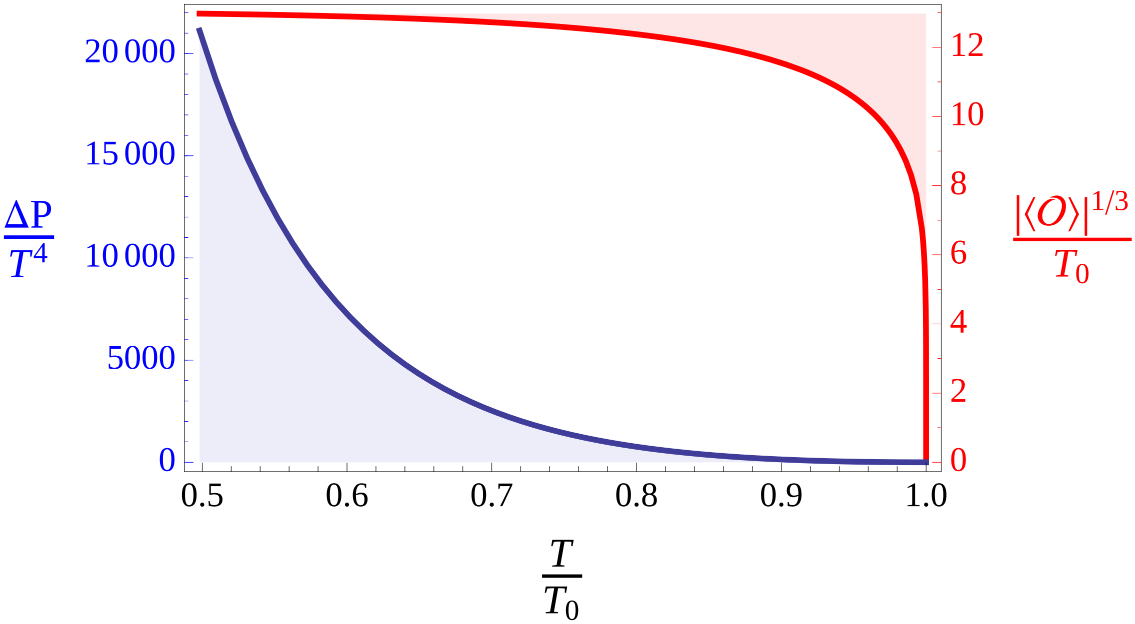



Figure 4: *Thermodynamic quantities for hairy BHs. The critical temperature is* $T_0 \approx 0.0607\mu$*. Near* $T_0$*,* $\langle \mathcal{O}_\psi \rangle \sim |T - T_0|^{1/2}$ *and* $\Delta P \sim (T - T_0)^2$*, indicating mean field critical exponents.*

# But what is the field theory operator that condenses?

- It's the lowest component of the F-term in the lagrangian:

$$\mathcal{O}_\psi = \sum_j \operatorname{tr} \lambda_{j\alpha}^2 + \mathcal{W}(\phi_i) \,, \tag{12}$$

  where $\lambda_{j\alpha}$ is the lowest component of the field strength for the jth node in the quiver, and $\mathcal{W}(\phi_i)$ is the lowest component of the superpotential $\mathcal{W}$.

- This operator is a chiral primary with R-charge $R = 2$ and conformal dimension $\Delta = 3$.

- As one decreases the field theory $T$ (at fixed $\mu$), the states with $\langle \mathcal{O}_\psi \rangle \neq 0$ start existing below a critical temperature $T_0$. At $T_0$, RNAdS becomes perturbatively unstable.

- For **a more general operator** $\mathcal{O}$ with R-charge $R$ and conformal dimension $\Delta$, the symmetry restored phase becomes perturbatively unstable at some $T_p(\Delta, R)$.
  - At $T = T_p$ the linearized eom for a scalar field with mass $m^2 = \Delta(4 - \Delta)$ in RNAdS background has a zero mode.

- Operators with $T_p > T_0$ (fixed $\mu$) are restricted to the shaded region of this plot:

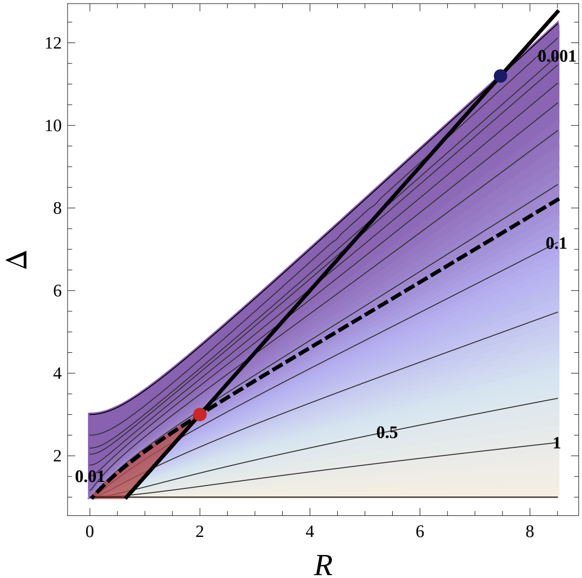
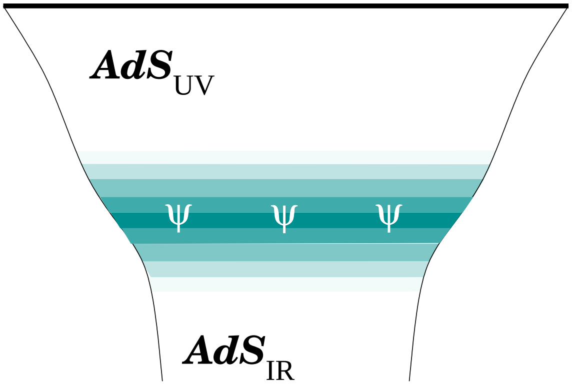



Figure 5: *A contour plot of $T_p/\mu$ as a function of conformal dimension $\Delta$ and R-charge $R$ of the more general field theory operator $\mathcal{O}$. Similar plot in [Denef, Hartnoll '09].*

- For $SE_5 = S^5$ or $T^{11}$, there are chiral primaries with lower conformal dimension. More about this later.

- For $SE_5 = S^5/\mathbf{Z}_7$ w/ weights $(1, 2, 4)$ $\mathcal{O}_\psi$ has lowest conformal dimension of all BPS operators. However, it is not the only operator with $\Delta = 3$ and $R = 2$.
  - Not clear whether this theory undergoes a superfluid phase transition at $T_p$, but it's possible.

What is the zero-temperature limit of the superconducting BHs where $\mathcal{O}_\psi$ condenses?




- It's an AdS-to-AdS domain wall solution [Gubser, SSP, Rocha '09]. See also [Gubser, Rocha '08].
- $F_{\mu\nu} \to 0$ in the IR, so the domain wall carries all the charge.
- If $SE_5 = S^5$, the IR geometry is an $SU(3)$-invariant non-SUSY vacuum.

## What is the conductivity at $T = 0$?

- At large $\omega$, $\mathrm{Re}\,\sigma \sim \omega$ (pure AdS result, dimensional analysis).
- At small $\omega$, $\mathrm{Re}\,\sigma \sim \omega^5$.
  - The small $\omega$ behavior is related to the dimension of the operator dual to $A_\mu$ in the IR and can be derived analytically. See [Gubser, SSP, Rocha '09].

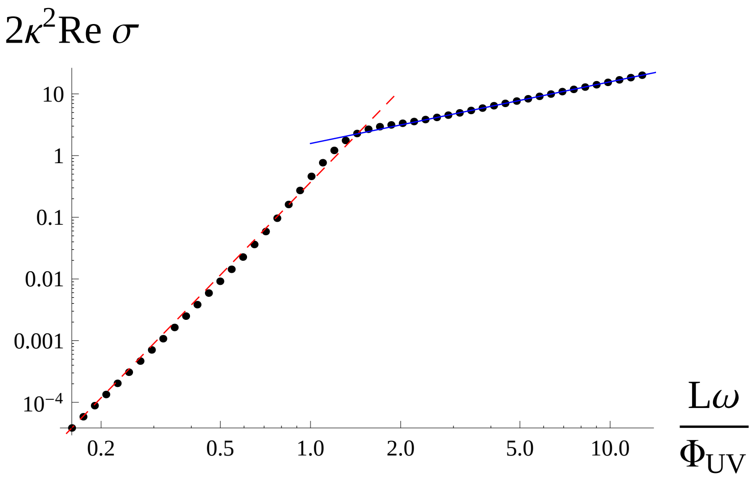



Figure 6: *Conductivity at* $T = 0$.

# 4. $\mathcal{N}=4$ SYM at finite $T$ and $\mu$

- The lagrangian of $\mathcal{N}=4$ SYM is

$$\mathcal{L}=\operatorname{tr}\Big\{-\frac{1}{2g_{\rm YM}^2}\mathcal{F}_{mn}\mathcal{F}^{mn}-(\mathcal{D}_mX_K)(\mathcal{D}^mX^K)+\frac{g_{\rm YM}^2}{2}[X_I,X_J]^2$$
$$+i\sum_a\lambda_a^\dagger\bar{\sigma}^m\mathcal{D}_m\lambda_a+g_{\rm YM}\sum_{a,b}C_{ab}^I\lambda_a[X_I,\lambda_b]+\bar{C}_{ab}^I\bar{\lambda}_a[X_I,\bar{\lambda}_b]\Big\}\,. \tag{13}$$

  Its field content is: a gauge field $\mathcal{A}_m$, six real scalar fields $X_I$ (transforming in the fundamental of $SO(6)$, and four Weyl fermions $\lambda_a$ (transforming in the fundamental of $SU(4)\simeq SO(6)$). All these fields are in the adjoint of the gauge group $SU(N)$.

- The vacuum state of $\mathcal{N}=4$ SYM corresponds to $AdS_5\times S^5$.

| $10$d type IIB SUGRA<br>$SO(6)$ realized<br>geometrically | $\Longrightarrow$ | $5$d AdS<br>*gauged* $SO(6)$ | $\Longrightarrow$ | $4$d $\mathcal{N}=4$ SYM<br>*global* $SO(6)$ |
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- The $U(1)_R$ described earlier is a subgroup of $SU(4) \simeq SO(6)$. Can choose it to be generated by $\text{diag}\{1,1,1,-3\}$ (for $SU(4)$), or equal phase rotations in the $12$-, $34$-, and $56$-planes (for $SO(6)$).

- The operator $\mathcal{O}_\psi$ described earlier is

$$\mathcal{O}_{\mathbf{10_C}} = \operatorname{tr}\left(\lambda_4\lambda_4 + Z_1[Z_2, Z_3]\right) \,, \tag{14}$$

where $Z_1 \equiv X_1 + iX_2$, $Z_2 \equiv X_3 + iX_4$, and $Z_4 \equiv X_5 + iX_6$.

  - it is part of the $\mathbf{10_C}$ of $SO(6)$), which contains operators of the schematic form $\operatorname{tr}\left(\lambda_a\lambda_b + X_I[X_J, X_K]\right)$.
  - it is a singlet under the $SU(3)$ commutant of the $U(1)$ above.

- The only other bosonic operator in $\mathcal{N}=4$ SYM with lower conformal dimension is

$$\mathcal{O}_{\mathbf{20}} = \operatorname{tr}(Z_1^2 + Z_2^2 + Z_3^2) \,. \tag{15}$$

  - it is part of the $\mathbf{20}$ of $SO(6)$, which contains operators of the schematic form $\operatorname{tr}\left(X_I X_J - \frac{1}{6}\delta_{IJ}X^K X_K\right)$.
  - under the $U(1)$ symmetry, $\mathcal{O}_{\mathbf{20}}$ has $2/3$ of the charge of $\mathcal{O}_{\mathbf{10_C}}$.

The embedding corresponding to $\mathcal{O}_{\mathbf{20}}$ was done in [Cvetic *et al.* '00]. The 5d effective lagrangian is in this case

$$\mathcal{L}_{\mathbf{20}} = R - \frac{3}{4}F_{\mu\nu}^2 - \frac{3}{2}\left[(\partial_\mu\eta)^2 + \sinh^2\eta\left(\partial_\mu\theta - \frac{2}{L}A_\mu\right)^2\right] + \frac{3}{L^2}(3 + \cosh(2\eta)) + (\text{Chern-Simons})\,. \tag{16}$$

- As described earlier, $\mathcal{L}_{\mathbf{10_C}}$ admits broken solutions *below* some critical temperature $T_{c,\mathbf{10_C}}$. These solutions are preferred over RNAdS.

- $\mathcal{L}_{\mathbf{20}}$ admits broken solutions *above* some critical temperature $T_{c,\mathbf{20}}$. These solutions are *not* preferred over RNAdS.

- Sadly, $T_{c,\mathbf{20}} > T_{c,\mathbf{10_C}}$.

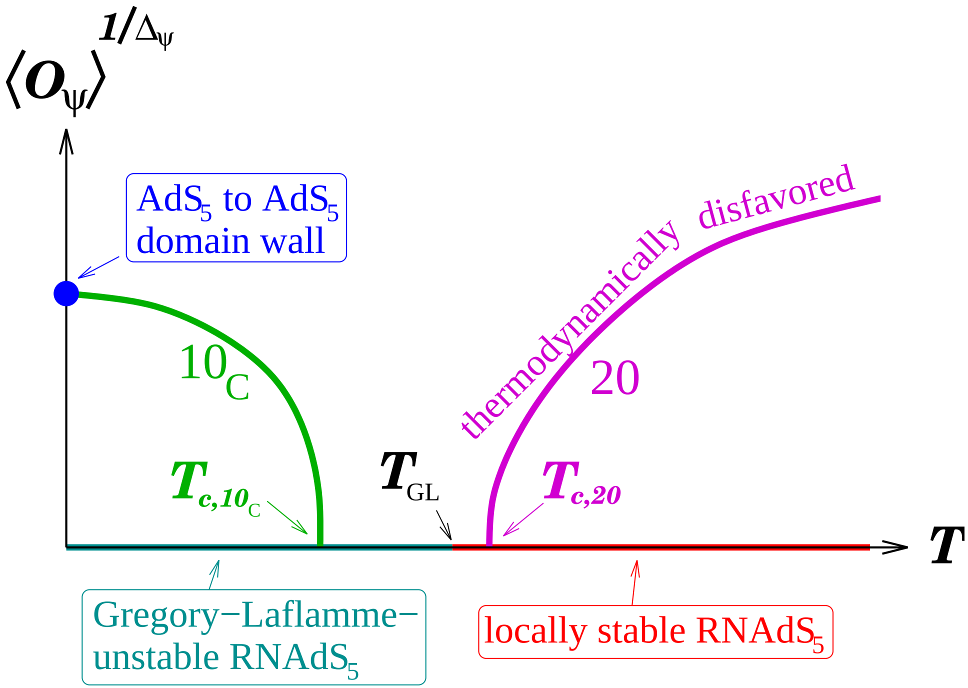


Figure 7: *Some known instabilities of spinning D3-branes, which correspond to $\mathcal{N} = 4$ SYM at finite temperature and R-charge chemical potential.*

# 5. Conclusions

- The Abelian Higgs model in AdS yields black hole solutions that break a gauge symmetry spontaneously.

- With some work, one can embed the Abelian Higgs model into superstring theory. There is a universal sector common to all theories dual to $AdS_5 \times SE_5$.

- Using AdS/CFT one can interpret the superconducting black holes as superfluid states in strongly coupled $\mathcal{N} = 1$ CFTs.

- There are still some puzzling aspects about the phase diagram of $\mathcal{N} = 4$ SYM. . .