Brownian Motion and AdS/CFT

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Munich, September 4, 2009

Based on last paper of 2008: arXiv:0812.5112
(JdB, Veronika Hubeny, Mukund Rangamani, Masaki Shigemori)
and work in progress

See also Son,Teaney, arXiv:0901.2338

OUTLINE

- Brownian motion
- AdS/CFT setup
- Hawking radiation and Brownian motion
- Fluctuation-Dissipation theorem
- Generalizations
- Mean free path
- Stretched horizon
- Conclusions

Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen

Einstein, 1905

Compare this to:

"How Bob Laughlin Tamed the Giant Graviton from Taub-NUT space"

Brownian Motion

Langevin equation:

$$\dot{p}(t) = -\gamma_0 p(t) + R(t)$$

friction

random force

$$\langle R(t) \rangle = 0$$

$$\langle R(t_1) R(t_2) \rangle = \kappa_0 \delta(t_1 - t_2)$$

Integrate:

$$p(t) = e^{-\gamma_0 t} \int dt' e^{\gamma_0 t'} R(t') dt'$$

Assume thermal equilibrium: $\langle m\dot{x}^2 \rangle = T$

Then

$$\langle [x(t) - x(0)]^2 \rangle = \frac{2D}{\gamma_0}(\gamma_0 t - 1 + e^{-\gamma_0 t})$$

$$\sim \frac{T}{m} t^2 \longrightarrow \sim 2Dt$$

$t \ll \gamma_0^{-1}$	$t = \gamma_0^{-1}$	$t \gg \gamma_0^{-1}$
ballistic regime	relaxation time	diffusive regime

$$D = \frac{T}{\gamma_0 m}$$

diffusion constant

(Einstein-Sutherland relation)

$$\gamma_0 = \frac{\kappa_0}{2mT}$$

fluctuation-dissipation theorem

This is too simple. Friction need not be instantaneous and random forces at different times need not be independent. Generalized Langevin equation:

$$\dot{p}(t) = -\int_0^t du\gamma(t-u)p(u) + R(t) + K(t)$$

memory kernel

$$\langle R(t)\rangle = 0$$

$$\langle R(t_1)R(t_2)\rangle = \kappa(t_1 - t_2)$$

external force

Fourier transform:

$$p(\omega) = \frac{R(\omega)+K(\omega)}{\gamma[\omega]-i\omega}$$

In particular

$$\langle p(\omega) \rangle = \frac{1}{\gamma[\omega]-i\omega} K(\omega)$$

admittance

Power spectrum:

$$\langle \mathcal{O}(\omega_1)\mathcal{O}(\omega_2) \rangle = 2\pi\delta(\omega_1+\omega_2) I_{\mathcal{O}}(\omega_1)$$

$$\Longrightarrow \quad I_p(\omega) = \frac{I_R(\omega)}{|\gamma[\omega]-i\omega|^2}$$

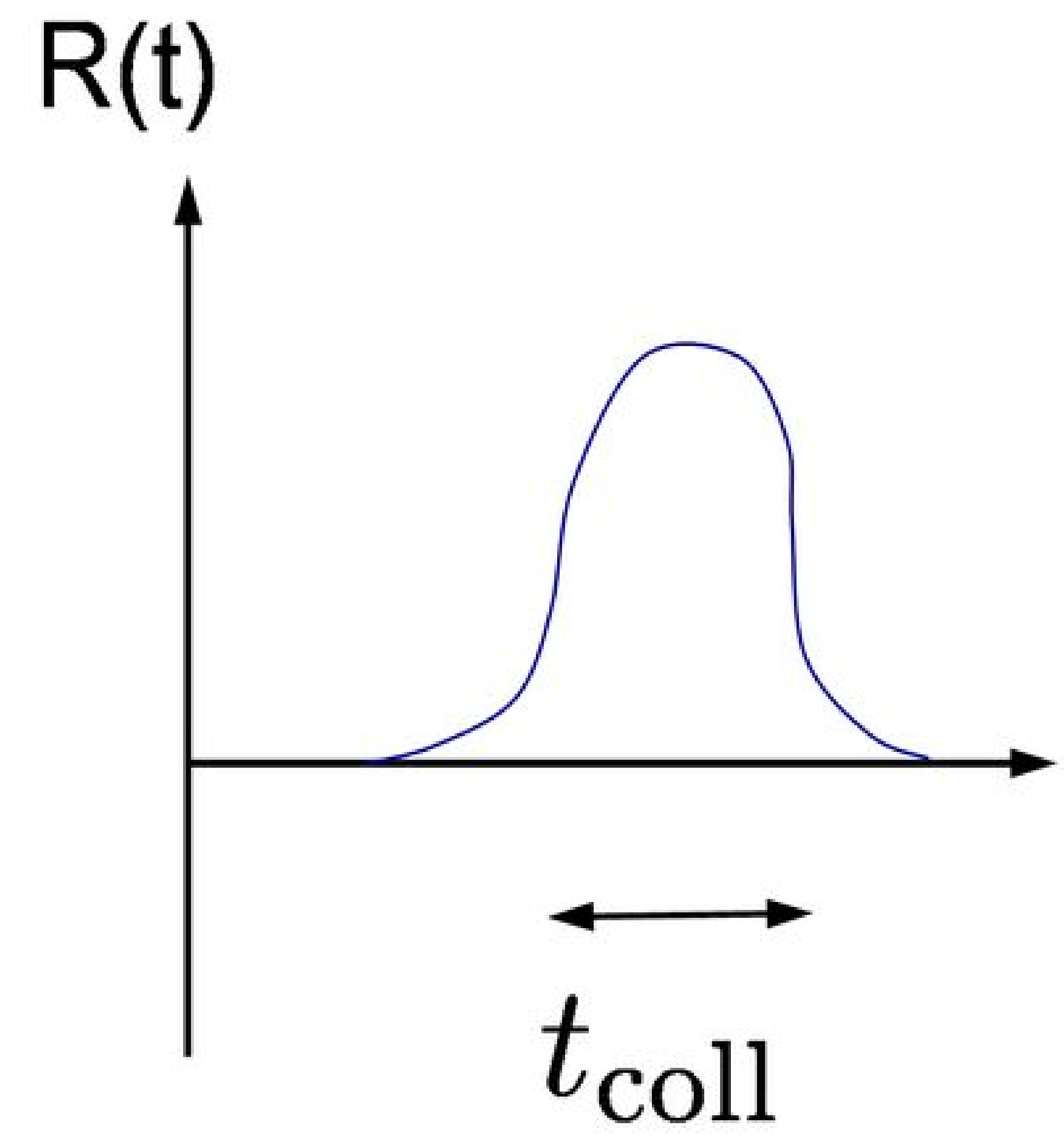
So in practice: first compute $\langle p(\omega) \rangle$ in the presence of an external force, then compute $\langle p(\omega_1) p(\omega_2) \rangle$ without an external force, and from this one can extract the two point function (power spectrum) of the random force.

Below we will exactly do this, but in the opposite order.

Define a relaxation time and a collision time as follows:

$$t_{\rm relax} = \left[\int_0^\infty dt\, \gamma(t)\right]^{-1} = \frac{1}{\gamma[\omega=0]}$$

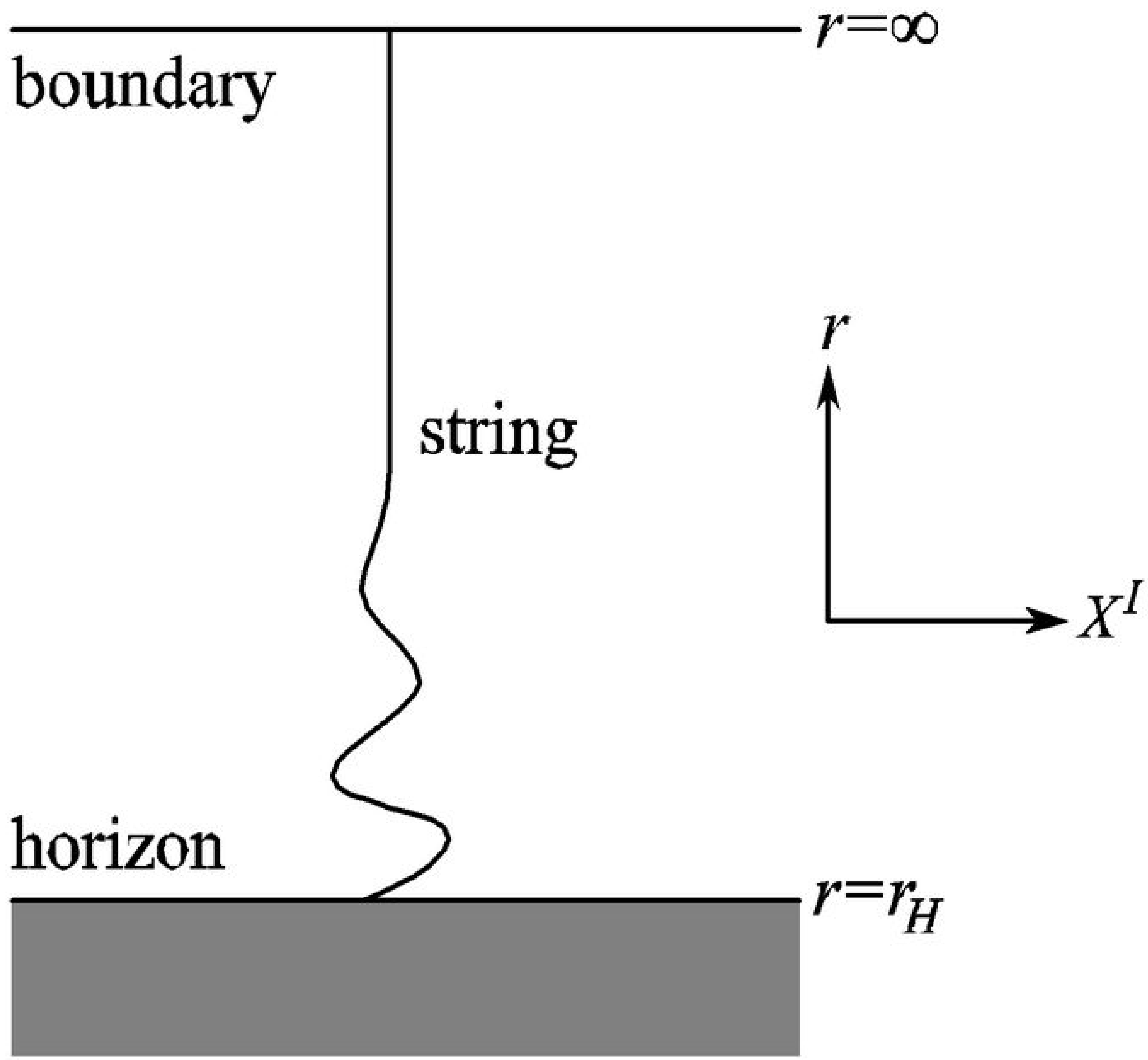
$$t_{\rm coll} = \int_0^\infty dt \frac{\kappa(t)}{\kappa(0)}$$

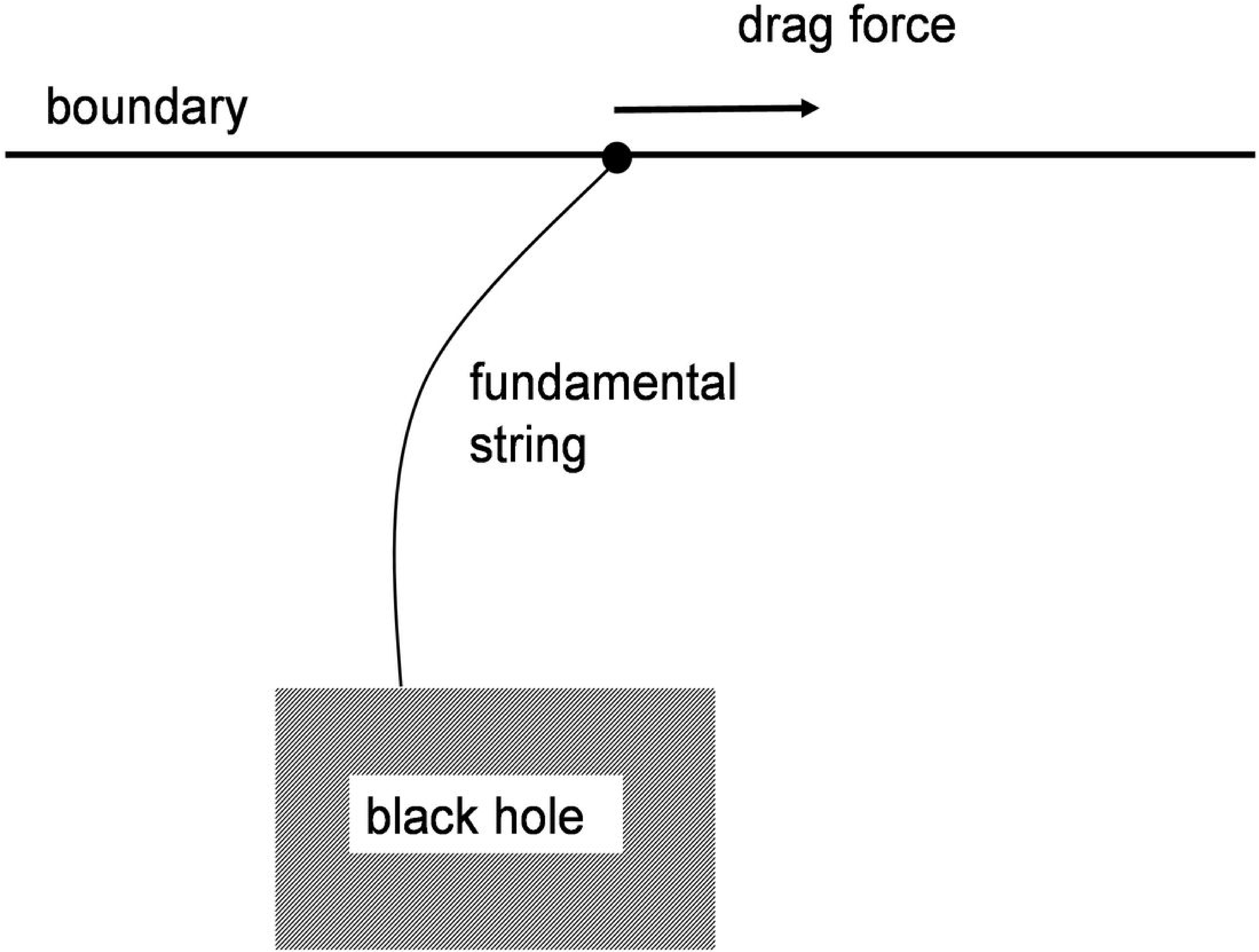


The collision time is the time scale over which collisions are correlated. Usually $t_{\rm relax} \gg t_{\rm coll}$.

This turns out not to be case in strongly coupled CFT plasma.

AdS/CFT setup




drag force
boundary
fundamental
string
black hole

The endpoint of a string describes a heavy probe particle (a "quark") in the strongly coupled CFT plasma.

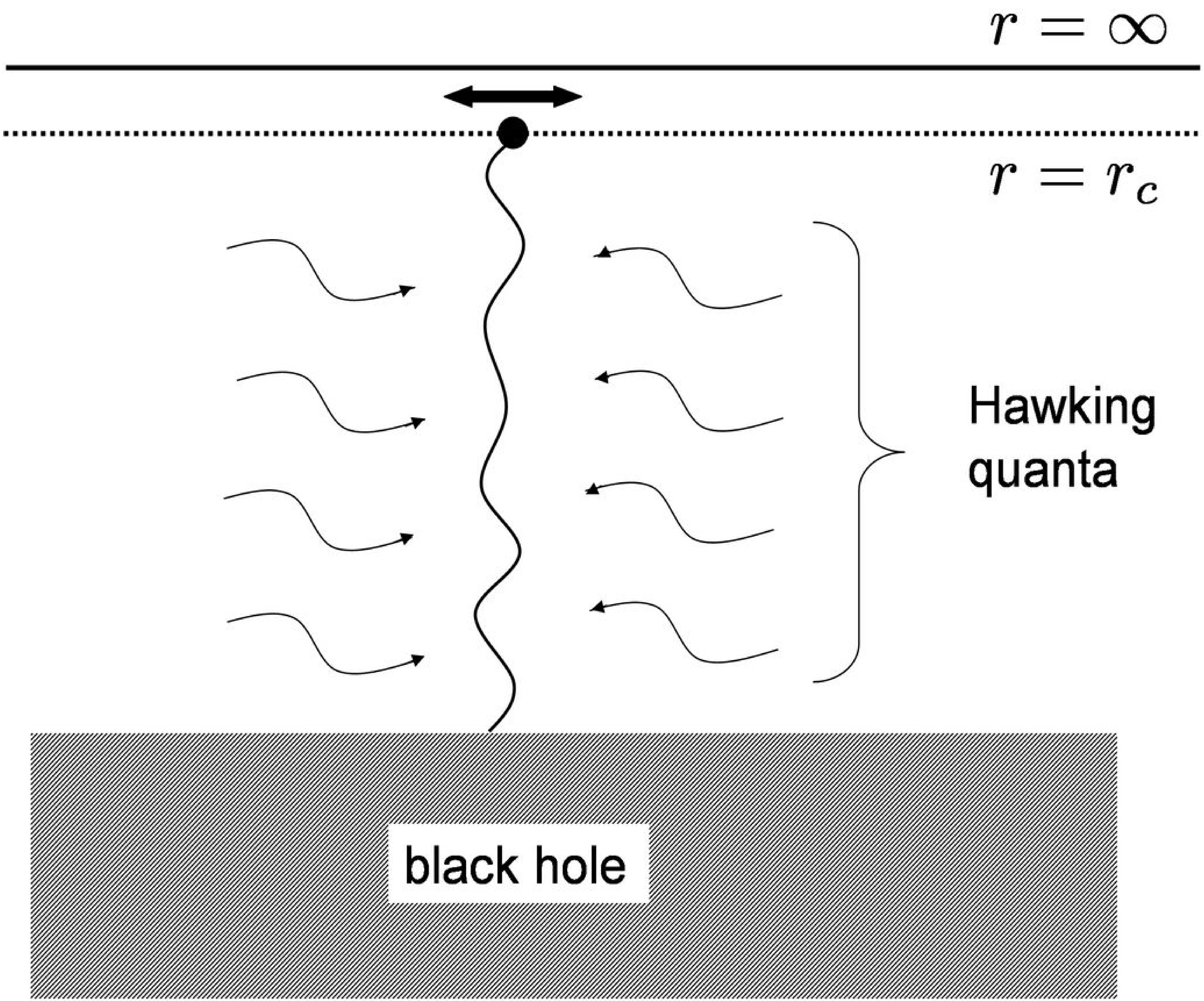
A similar setup was used to compute the drag force on a quark in the quark-gluon plasma.

Herzog, Karch, Kovtun, Kozcaz, Yaffe; Gubser

Such a quark at finite temperature is in particular expected to undergo Brownian motion

What is the bulk description of Brownian motion?

The black hole environment must excite the string in such a way that the endpoint undergoes Brownian motion



$r = \infty$
$r = r_c$
Hawking
quanta
black hole

We therefore take a fundamental string in an AdS Schwarzschild background

$$ds_d^2 = \frac{r^2}{\ell^2}\left[-h(r)dt^2 + dX^i dX^i\right] + \frac{\ell^2}{r^2 h(r)} dr^2$$

with $h(r) = 1 - \left(\frac{r}{r_H}\right)^{d-1}$

Describe string via Nambu-Goto action, go to static gauge and expand to second order: free 2d theory for the scalar fields X.

Explicitly:

$$S_{\rm NG} \sim \int dtdr \left[-\det \begin{pmatrix} -\frac{h(r)r^2}{\ell^2} + \frac{r^2}{\ell^2}\dot{X}\dot{X} & \frac{r^2}{\ell^2}\dot{X}X' \\ \frac{r^2}{\ell^2}\dot{X}X' & \frac{\ell^2}{r^2 h(r)} + \frac{r^2}{\ell^2}X'X' \end{pmatrix} \right]^{1/2}$$

$$\sim \int dtdr \left[-\frac{1}{h(r)}\dot{X}\dot{X} + \frac{r^4 h(r)}{\ell^4} X'X' \right]$$

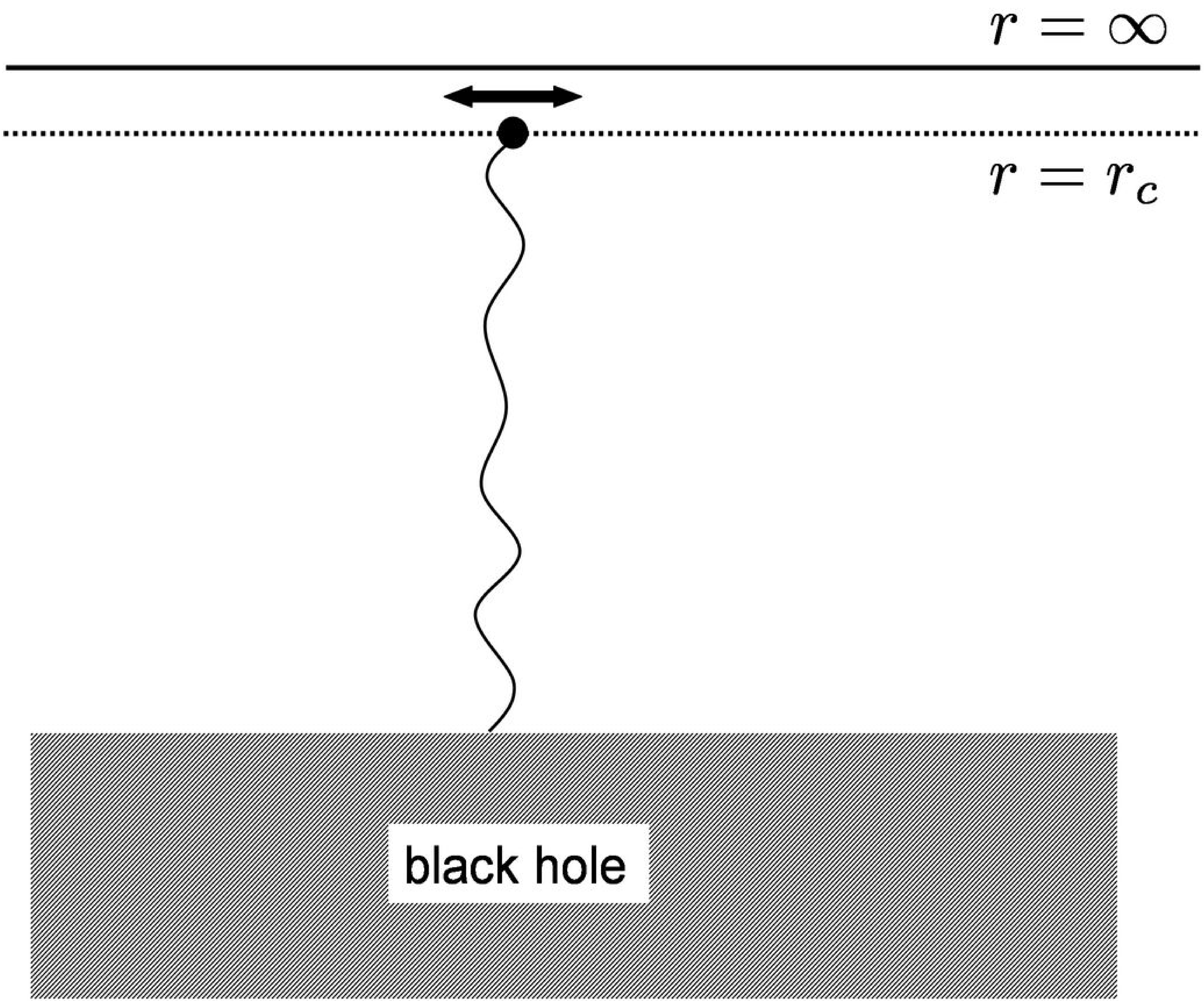
$$\longrightarrow \left[-\partial_t^2 + \frac{h(r)}{\ell^4}\partial_r\left(r^4 h(r)\partial_r\right)\right] X(t,r) = 0$$

Boundary conditions??

Impose UV cutoff at $r=r_c$ and Neumann boundary conditions there: quark can move freely. Mass

$$m \sim \frac{r_c - r_H}{2\pi\alpha\prime}$$

In practice also introduce an IR cutoff. If one takes the IR cutoff to zero the results remain finite. But see later.



$r = \infty$
$r = r_c$
black hole

Solution to the wave equation:

$$X(t,r) = \sum_\omega (a_\omega u_\omega(t,r) + \text{c.c.})$$

with

$$u_\omega(t,r) \sim [f_\omega^+(r) + B f_\omega^-(r)] e^{-it}$$

$$[a_\omega, a_{\omega\prime}^\dagger] = \delta_{\omega,\omega\prime}$$

outgoing

infalling

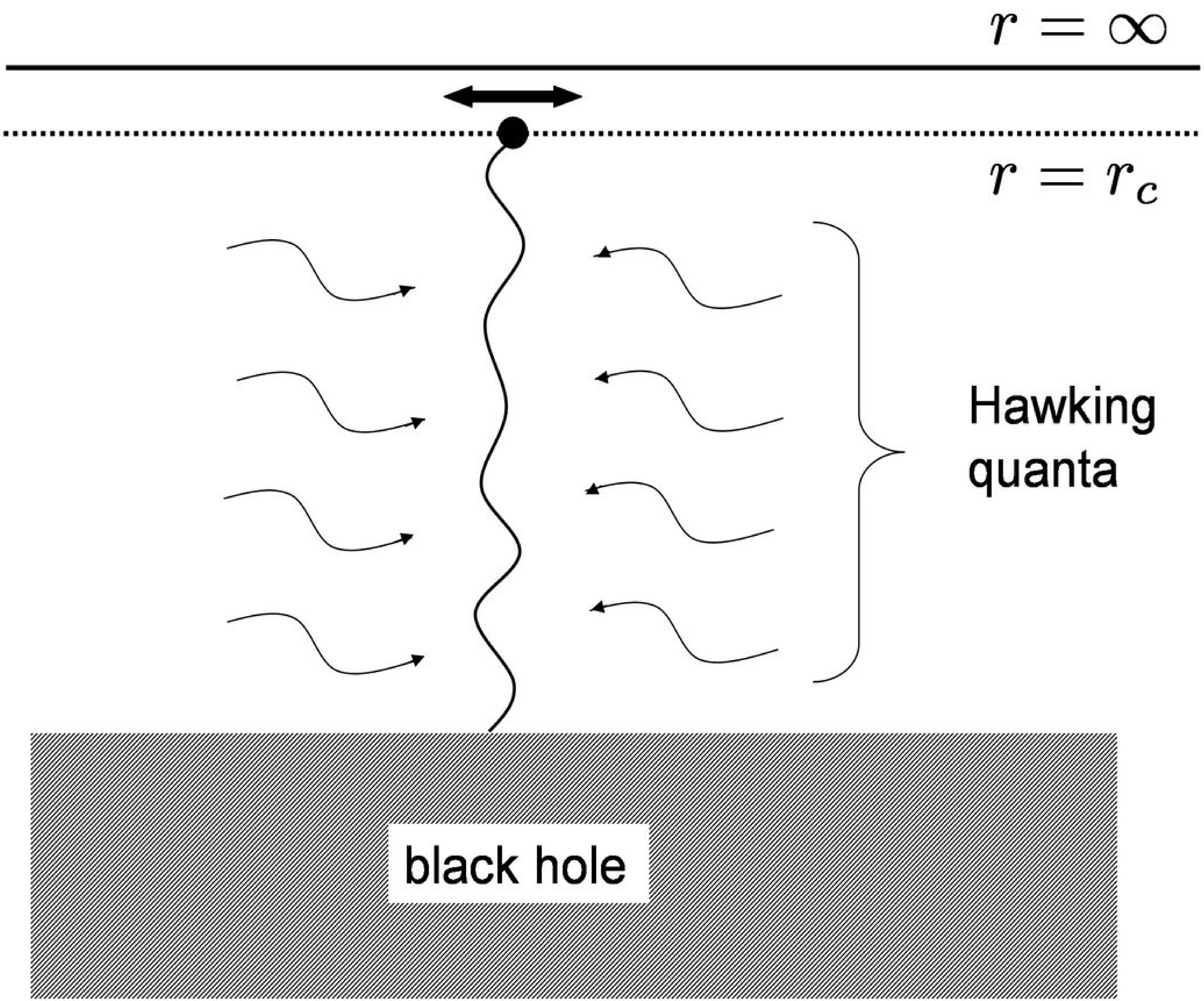
at the horizon

Hawking quanta kick the string: at the horizon the emerging modes should be thermally occupied:

$$\langle a^\dagger_\omega a_{\omega'} \rangle = \frac{\delta_{\omega,\omega'}}{e^{\beta\omega}-1}$$

result (d=3):

$$\langle : x(t)^2 : \rangle = \frac{4\alpha'}{\pi^2 T^2 \ell^2} \int_0^\infty \frac{d\omega}{\omega} \frac{1+\frac{\omega^2 \ell^4}{r_h^2}}{1+\frac{\omega^2 r_c^2 \ell^4}{r_h^4}} \frac{\sin^2(\omega t/2)}{e^{\beta\omega}-1}$$


$r = \infty$
$r = r_c$
Hawking quanta
black hole

$$\sim \frac{T}{m} t^2 \longrightarrow \sim 2Dt$$

$t \ll t_c$	$t = t_c$	$t \gg t_c$
ballistic regime	relaxation time	diffusive regime

where:

$$t_c = \frac{\alpha' m}{\ell^2 T^2}$$

$$D = \frac{\alpha'}{2\pi \ell^2 T}$$

Intuition: $s \sim \sqrt{tT}\ \ T^{-1}$

T^{-1} → mean free path

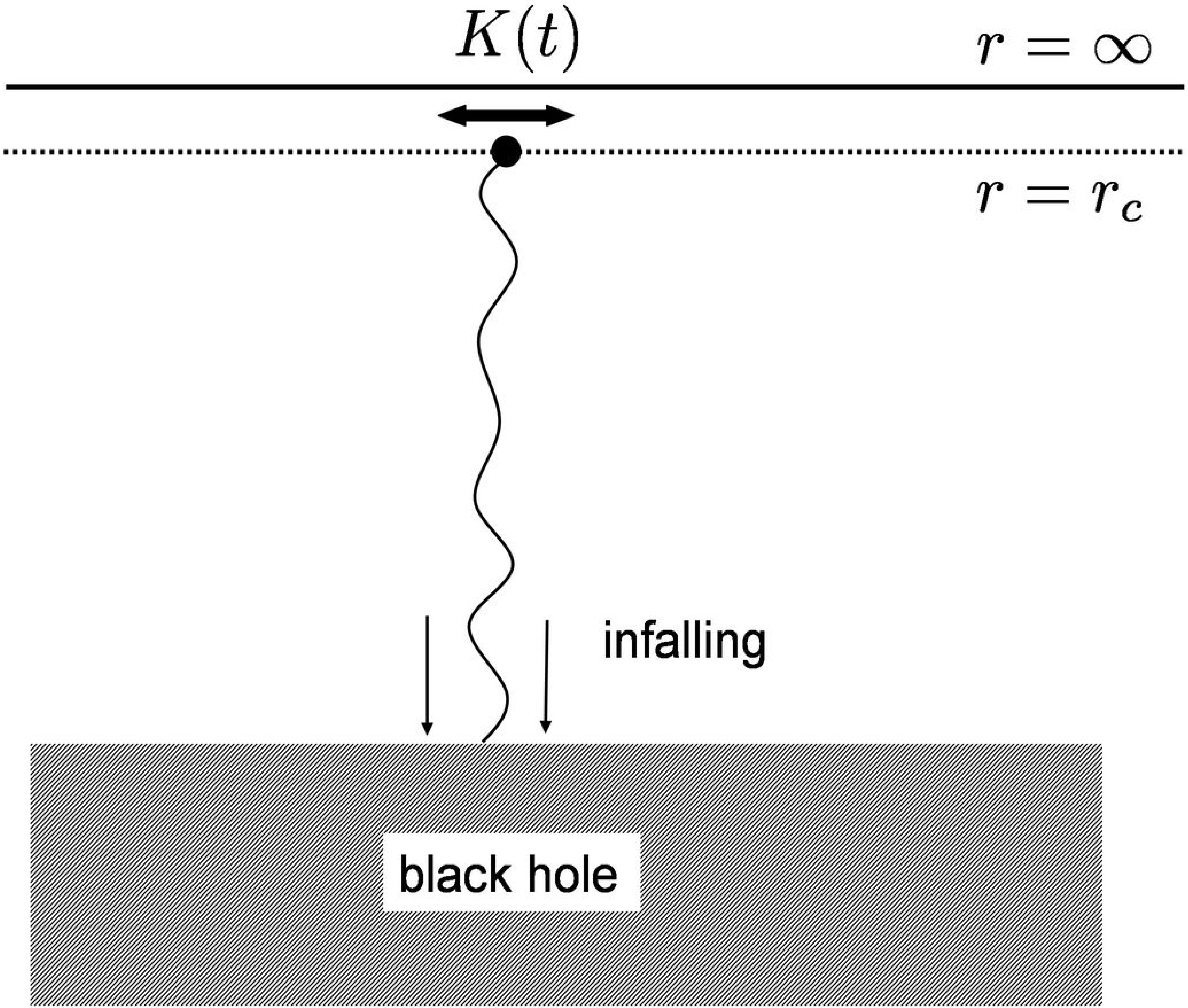
tT → Number of collisions

$$\langle p(\omega)\rangle = \frac{1}{\gamma[\omega]-i\omega}K(\omega)$$

To find admittance, consider forced motion: apply an electric field at the boundary and solve equations for the string. Relevant boundary conditions: purely infalling boundary conditions at the horizon. (reason: need retarded propagator).

Result

$$\frac{1}{\gamma[\omega]-i\omega} = \frac{\alpha' m}{2\pi\ell^2 T^2}\frac{1-\frac{i\omega\ell^2}{r_c}}{1-\frac{i\omega\ell^2 r_c}{r_h^2}}$$


$K(t)$
$r = \infty$
$r = r_c$
infalling
black hole

$$I_p(\omega) = \frac{I_R(\omega)}{|\gamma[\omega] - i\omega|^2}$$

Can now determine the nature of the random force:

$$I_R(\omega) = \frac{4\pi T^3 \ell^2}{\alpha'} \frac{1 + \frac{\omega^2 \ell^4}{r_h^2}}{1 + \frac{\omega^2 r_c^2 \ell^4}{r_h^4}} \frac{\beta|\omega|}{e^{\beta|\omega|} - 1}$$

From this obtain $t_{\mathrm{coll}} \sim \frac{1}{T}$

Recall $t_c = \frac{\alpha' m}{\ell^2 T^2}$

$$\Longrightarrow \quad \frac{t_c}{t_{\rm coll}} = \frac{\alpha' m}{\ell^2 T} \sim \frac{m}{\sqrt{\lambda} T}$$

If $T \ll m \ll \sqrt{\lambda} T$ then the thermalization time is much less than the collision time, nevertheless the quark is much heavier than the typical energy 1/T of a fluid particle..

Apparently many collisions take place within the time it takes for a single collision to take place? Come back to this later.

Notice that we did two separate computations. One with an external force and infalling boundary conditions (which gave the admittance), and one with Neumann boundary conditions (which gave the power spectrum). These two computations should not be independent due to the fluctuation dissipation theorem.

Suppose a system is in thermal equilibrium and we add to the Hamiltonian a perturbation –AK(t). Then to first order, for any quantity B

$$\Delta B(t) = -i \int_{-\infty}^{t} dt' \langle [A(0), B(t-t')] \rangle_T$$

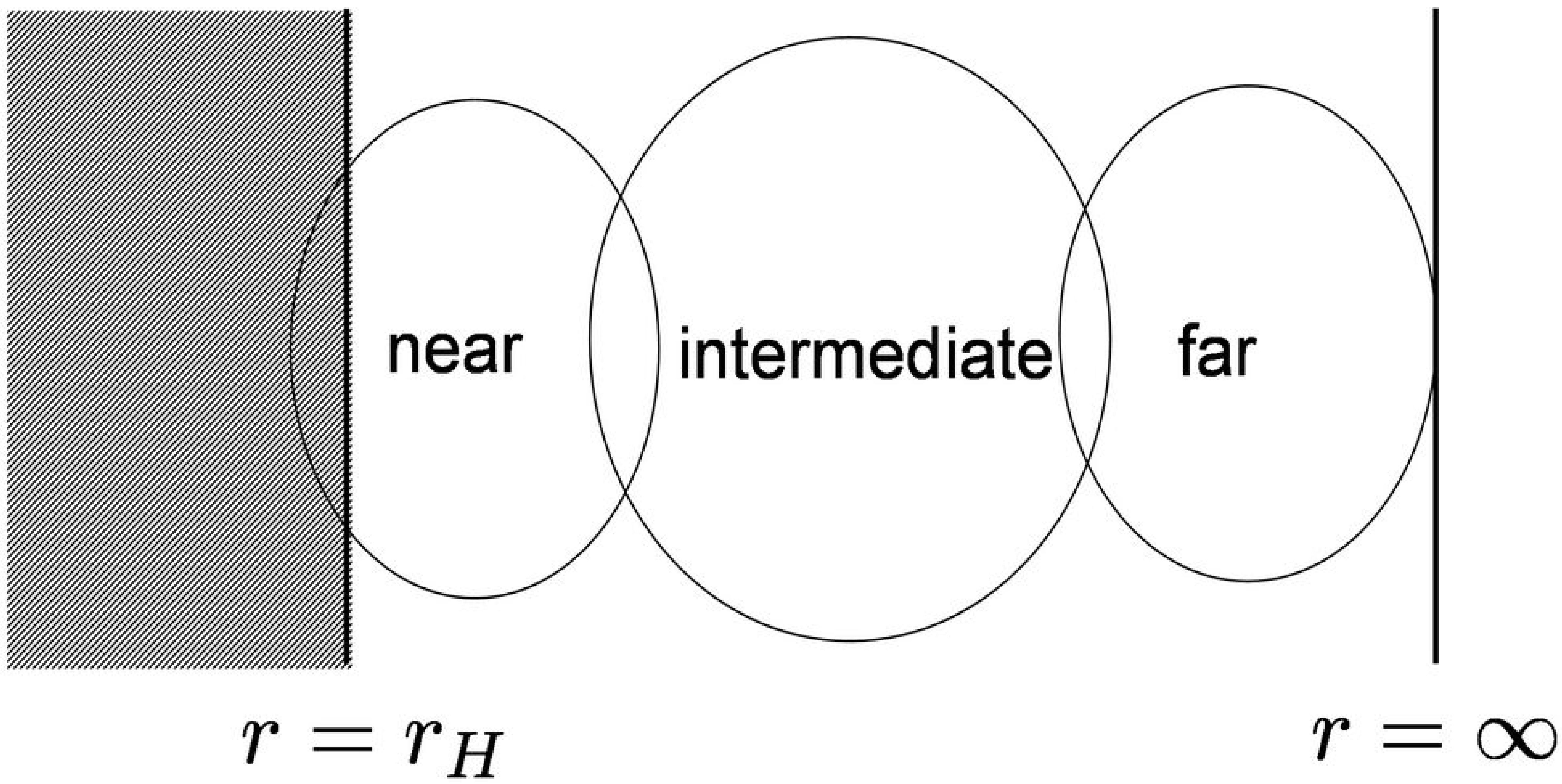
(Kubo formula)

One can show that in particular this implies

$$2\text{Re}\ \gamma(\omega) = \frac{1}{mT} I_R(\omega)$$

The results of the bulk computation indeed satisfy this relation. One can even prove this relation directly from the bulk point of view.

Everything so far worked only for d=3. For higher d, we cannot solve the equations of motion exactly and we have to rely on approximate methods.



One can solve the wave equation approximately in each of the three regions and match these solutions to each other.

In particular, for zero frequency we obtain

$$\gamma[\omega=0]=\frac{8\pi\ell^2T^2}{(d-1)^2\alpha' m}$$

which agrees with previous results (Herzog et al, Gubser).

Mean free path

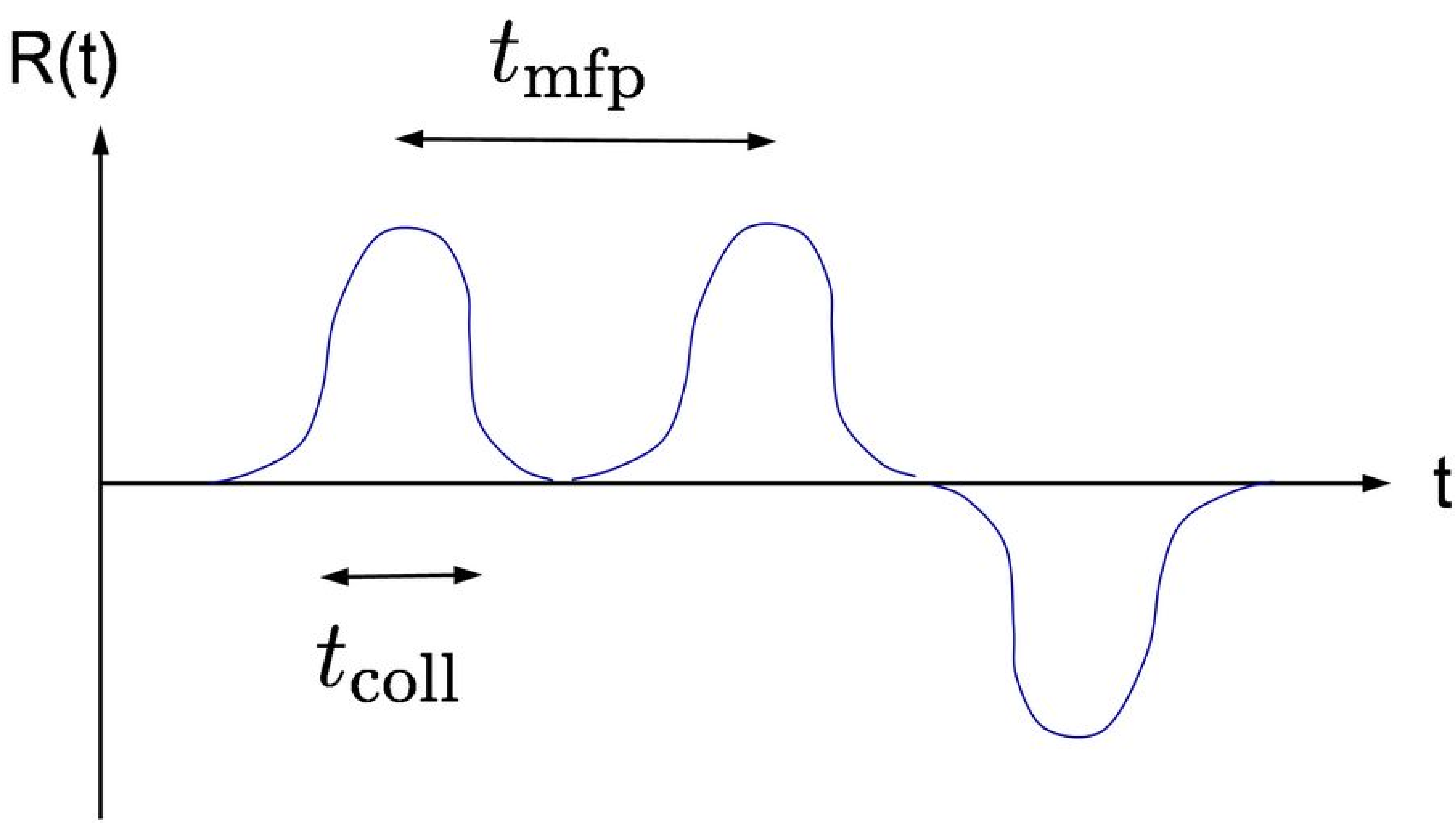
Recall $$\frac{t_c}{t_{\rm coll}} = \frac{\alpha' m}{\ell^2 T} \sim \frac{m}{\sqrt{\lambda} T}$$

In standard kinetic theory, we expect

$$t_{mfp} \gg t_{coll}$$

but then the thermalization time should also be much longer than the collision time, which need not be the case. Can we estimate the mean free path time?

Imagine the random force to be sequence of pulses with random signs. Due to the random sign, the two-point correlator of R(t) does not see the mean free path.



In order to measure the mean free path, we need to consider the four-point function of R(t). This requires an expansion of the Nambu-Goto action to fourth order.

One can then compute the four-point function. It is both UV and IR divergent. The UV divergence can be canceled using holographic renormalization.

The interpretation of the IR divergence is unclear right now. The retarded four-point function is IR finite but that does not seem to be the right physical quantity to compute.

Breakdown of random collision picture?

Naively dropping the IR divergence leads to a finite four point correlator.

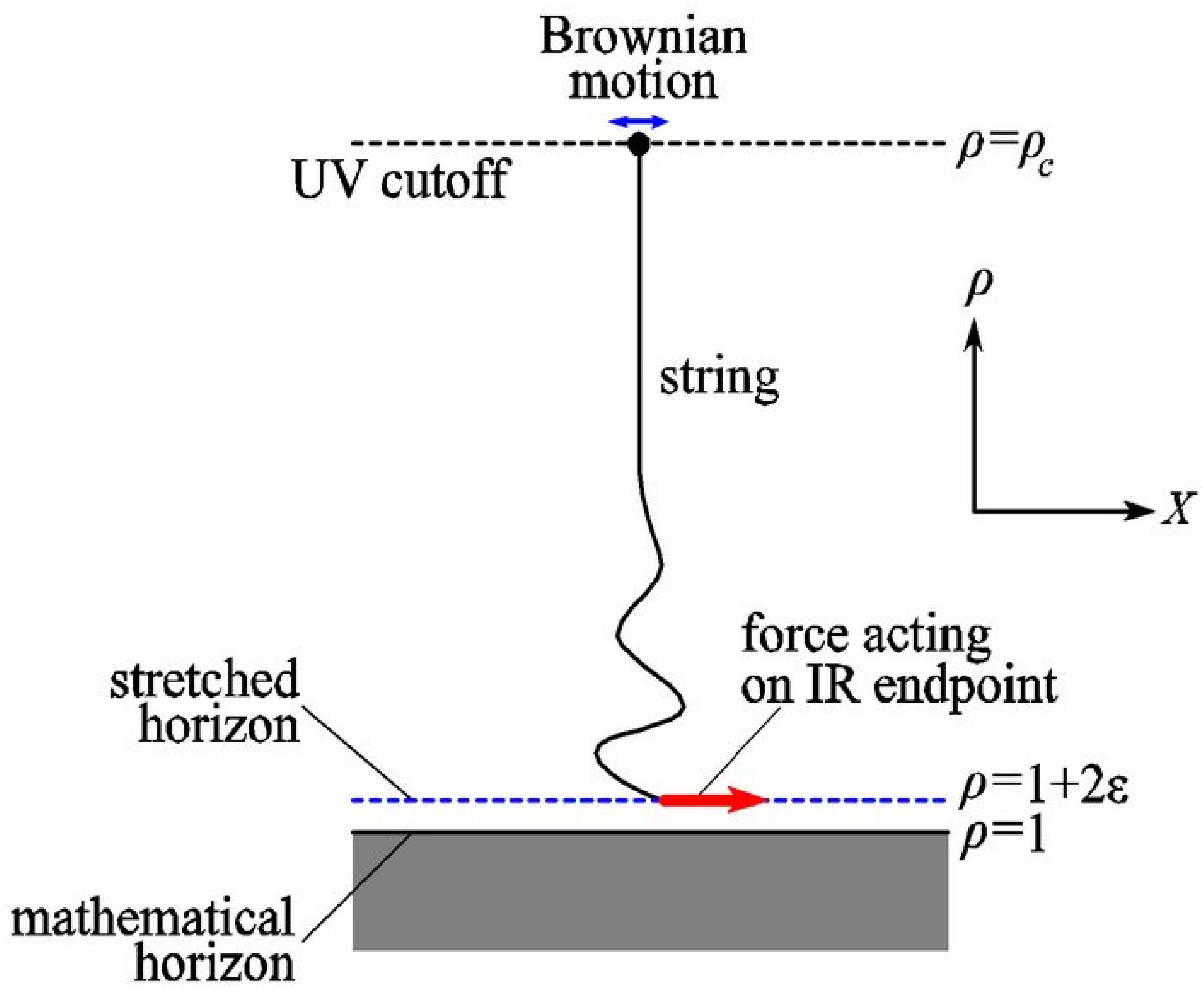
Extract t_{mfp} from four point correlator of random force. Result:

$$t_{\mathrm{mfp}} \sim \frac{1}{\sqrt{\lambda} T} \ll t_{\mathrm{coll}} \sim \frac{1}{T}$$

Standard picture of Brownian motion is not valid at strong coupling. Perhaps this is not too surprising, since standard kinetic theory which is valid at weak coupling also predicts a large viscosity to entropy ratio.

The membrane paradigm and Brownian motion

Basic idea: physics can be effectively described by an imaginary membrane equipped with physical properties which is sitting just outside the horizon.



Suppose the endpoint of the string that lives on the membrane experiences friction and a random force. When does this reproduce the Brownian motion of the UV endpoint??

$$-\frac{2r_H^4\epsilon}{\pi\alpha'\ell^4}\partial_r X = -\frac{2\pi\ell^2}{\alpha'\beta^2}\partial_t X + \tilde{R}(t)$$

friction term cancels infalling modes

random force

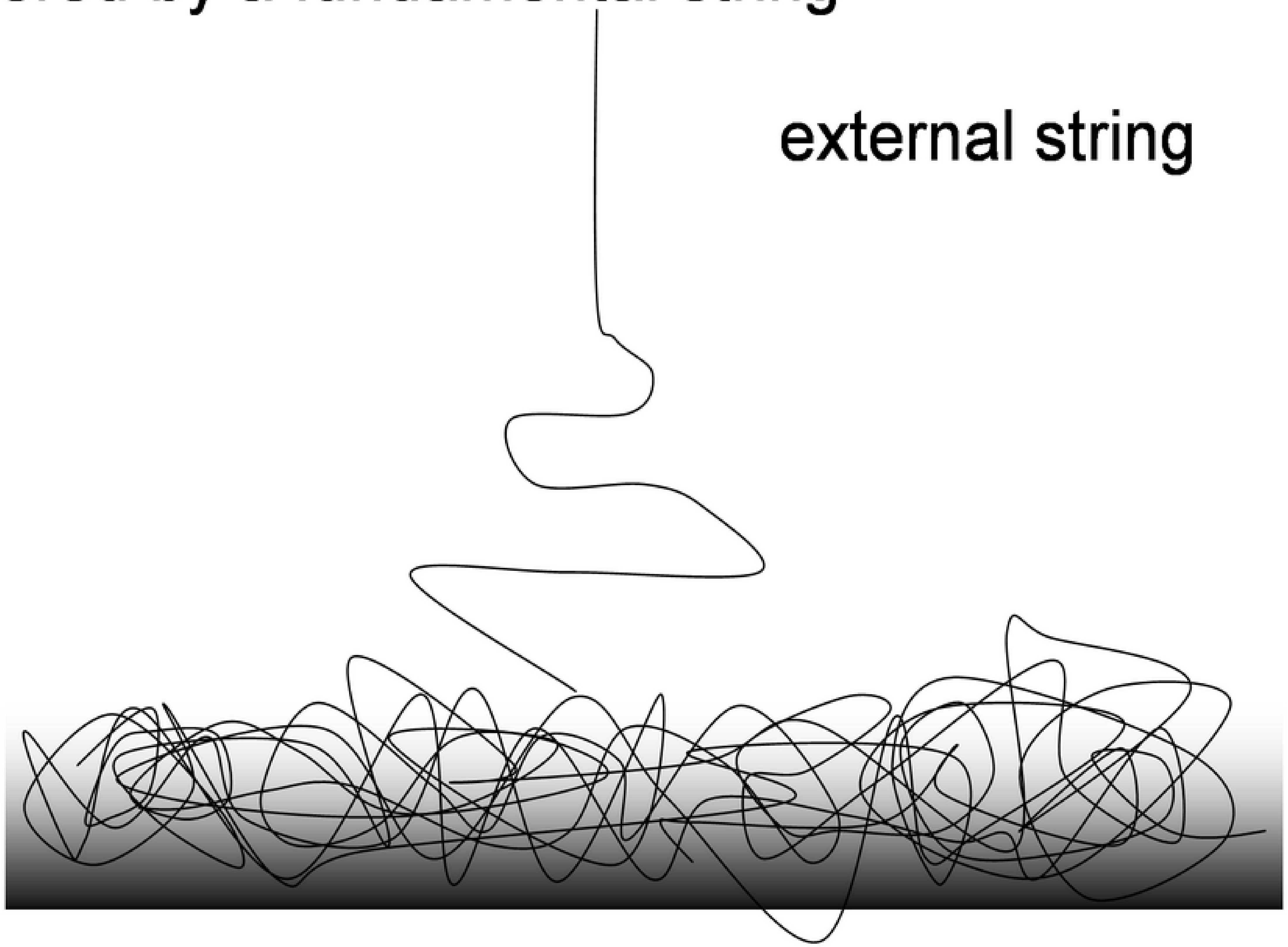
$$\langle \tilde{R}(t)\tilde{R}(t')\rangle \approx \frac{\pi\ell^2}{\alpha'\beta^3}\delta(t-t')$$

thermally occupies outgoing modes

Diffusion constant obtained from this is the same as the diffusion constant for the quark in the boundary theory: example of a universal feature derived from the membrane paradigm (cf Iqbal, Liu).

Similar reasoning can be applied to η/s.

Related idea: horizon of the black hole is covered by a fundamental string



Stretched horizon sits a distance ℓ_s away from actual horizon. It is occupied by a string of length $L \sim S\ell_s$ with S the entropy of the black hole.

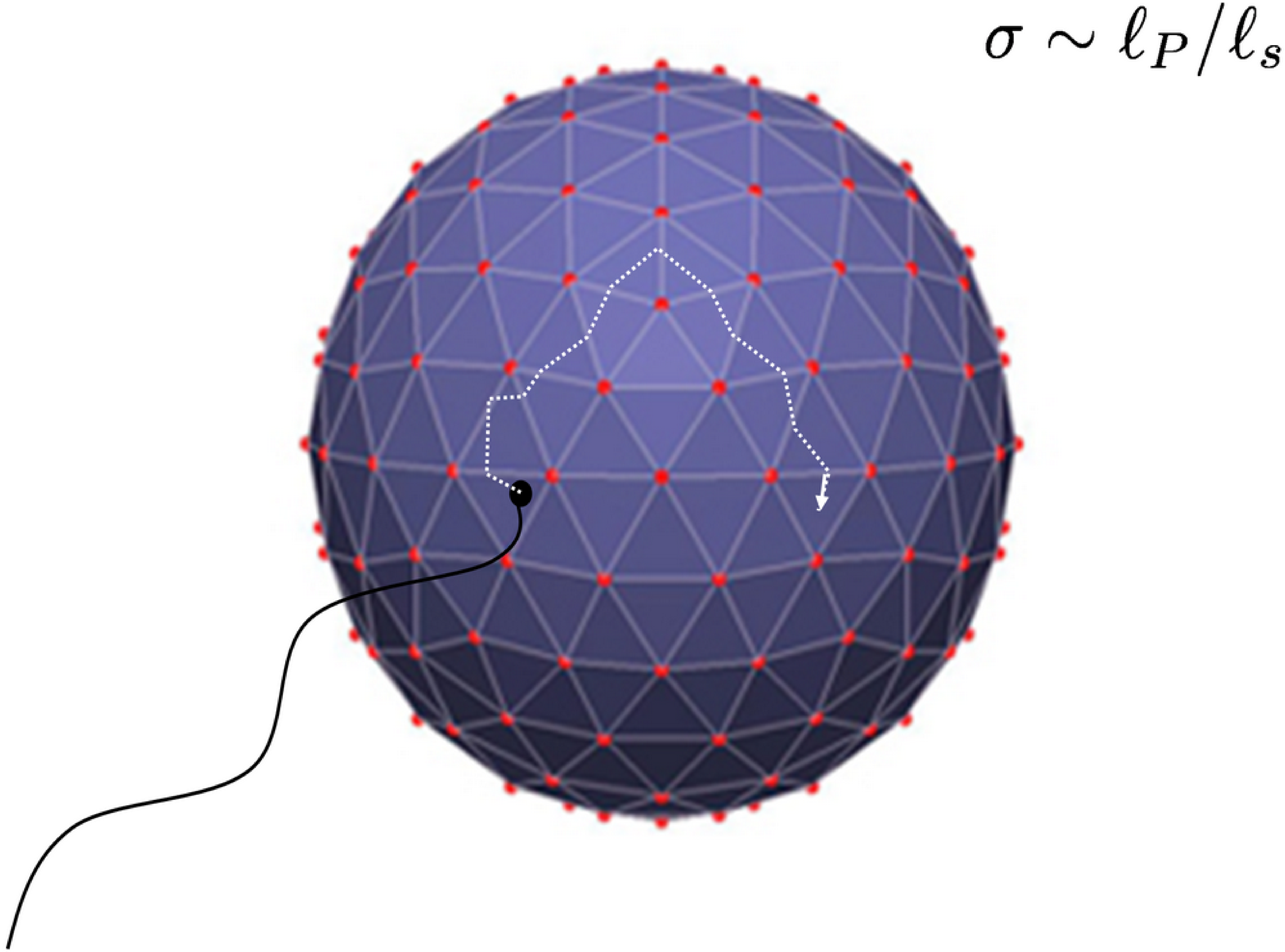
We assign one degree of freedom to each segment of length ℓ_s . Average separation between quasi-particles is

$$\Delta X \sim \frac{\ell}{r_H} G_d^{1/(d-2)}$$

If quasi-particles move with the speed of light, get mean free path

$$t_{\rm mfp} \sim \Delta X \ell / \sigma \ell_s$$

with σ the scattering probability. On dimensional grounds expect $t_{\rm mfp} \sim 1/T$ which then leads to $\boxed{\sigma \sim \ell_P / \ell_s}$???


$\sigma \sim \ell_P/\ell_s$

Conclusions

- New probe of black hole physics/more detailed properties of the quark-gluon plasma.
- Found the admittance and power spectrum, collision and relaxation timescales, and verified the fluctuation-dissipation theorem.
- Suggestive (but unreliable) evidence for the stretched horizon picture (ignored backreaction and higher order terms in the action)
- Would be interesting to generalize the analysis to the relativistic case.

- A very precise measurement of Brownian motion could in principle reveal correlations in Hawking radiation!
- Clarify nature of IR divergence of four-point function.
- To do: elucidate nature of quasiparticles that occupy the stretched horizon.
- Compare and contrast to other approaches to black hole physics. Evidence for very fast scrambling of information?