# Relaxation time of Holographic Plasma

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Based on: arXiv:0908.0108 and other work

## Motivation:

$\Rightarrow$ There was been many studies were the gauge/string theory correspondence framework was been used to extract transport coefficients of strongly coupled gauge theory plasma.

*however...*

$\Rightarrow$ real QCD is not in any one of the models studied

(it is possible to reach QCD as a particular limit in some of the models, but the price to pay is too big: the truncation of the full string theory to a supergravity sector is inconsistent)

*thus...*

$\Rightarrow$ one attempts to discover common/universal features of hydrodynamics of strongly coupled gauge theories (by looking at the explicit string theory models as well as phenomenological models) and

*hope...*

$\Rightarrow$ that QCD is in the universality class of the models studied

Examples:

- the shear viscosity ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

- the bulk viscosity ratio

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right), \qquad c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}$$

$\Rightarrow$ there is one other important transport coefficient, the (effective) relaxation time $\tau_{eff}$. Can some universal equality/inequality be established for it?

## Now my real motivation:

$\Rightarrow$ The was an interesting note (arXiv:0907.2262) co-authored by one of the leaders in hydrodynamic simulations of sQGP, Ulrich Heinz. They point out that shear viscosity suppresses the elliptic flow in RHIC collisions, but so does the bulk viscosity. The reason for this is very simple. In the relevant hydro simulations the combination of the transport coefficient that enters is

$$\eta_{CFT} \to \eta_{eff} = \eta \left(1 + \frac{3\zeta}{4\eta}\right)$$

$\Rightarrow$Now, bulk viscosity can be rather large for off-center RHIC collisions. Let me explain why. We start with the lattice data for the QCD equation of state:

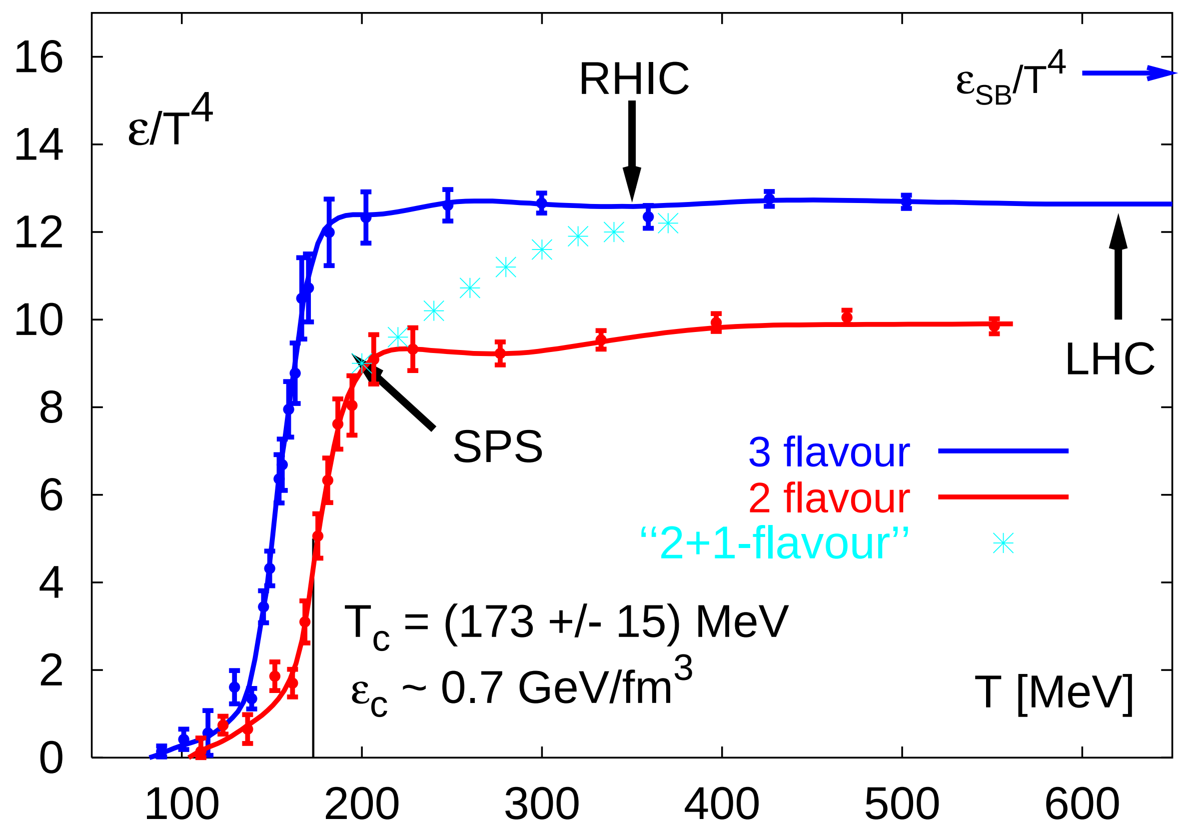


Figure 1: QCD thermodynamics from lattice; F.Karsch and E.Laermann, hep-lat/0305025. The RHIC arrow corresponds to $\frac{T}{T_c} \sim 1.8$. However to off-central collisions it can drop to $\frac{T}{T_c} \sim 1.4$ ($\sim 240$ MeV).

$\Rightarrow$ The message: we are rather close to a phase transition

$\Rightarrow$ We need to look at the holographic plasma model, that shares similarities with the QCD equation of state, and try to extract the information from that model about the bulk viscosity

$\Rightarrow$ My favorite model for such analysis is the $\mathcal{N}=2^*$ plasma. You should think of it as a particular mass deformation of the $\mathcal{N}=4\ SU(N)$ plasma. The mass parameter, $m$, is responsible for a nontrivial phase transition in this model. The phase transition that occurs in this model is in the universality class of the mean field tricritical point,

$$c_V \sim |1 - T_c/T|^{-1/2}$$

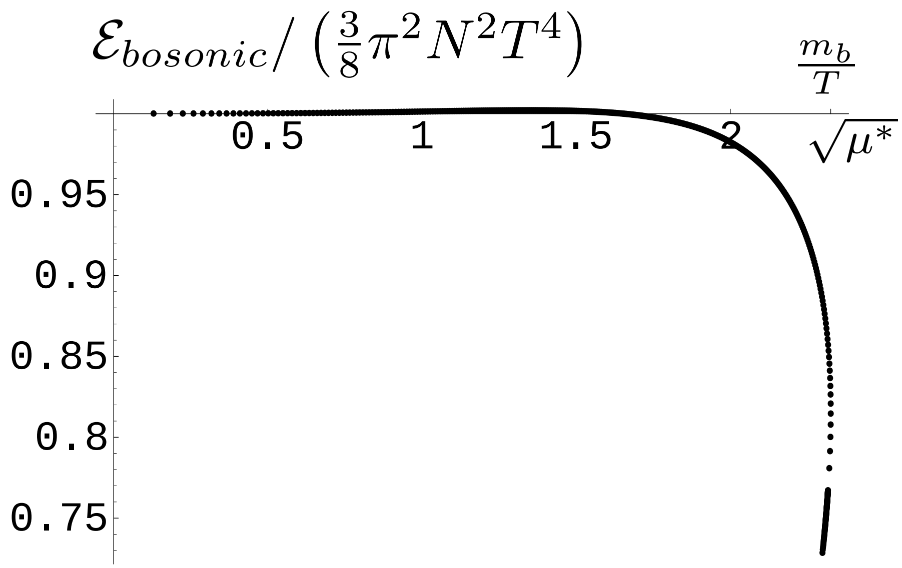


Figure 2: Equation of state of the mass deformed $\mathcal{N}=4$ gauge theory plasma. Note that $\frac{m_b}{T_c}\approx 2.29$

In this model:

$$\text{RHIC}: \ \frac{T}{T_c}\sim 1.8 \qquad \Leftrightarrow \qquad \frac{m_b}{T}\sim 1.27 \qquad \Rightarrow \left|1-\frac{\mathcal{E}_{bosonic}}{\mathcal{E}_{CFT}}\right| \ll 1$$

$\Rightarrow$ What is the bulk viscosity near the phase transition in this model?

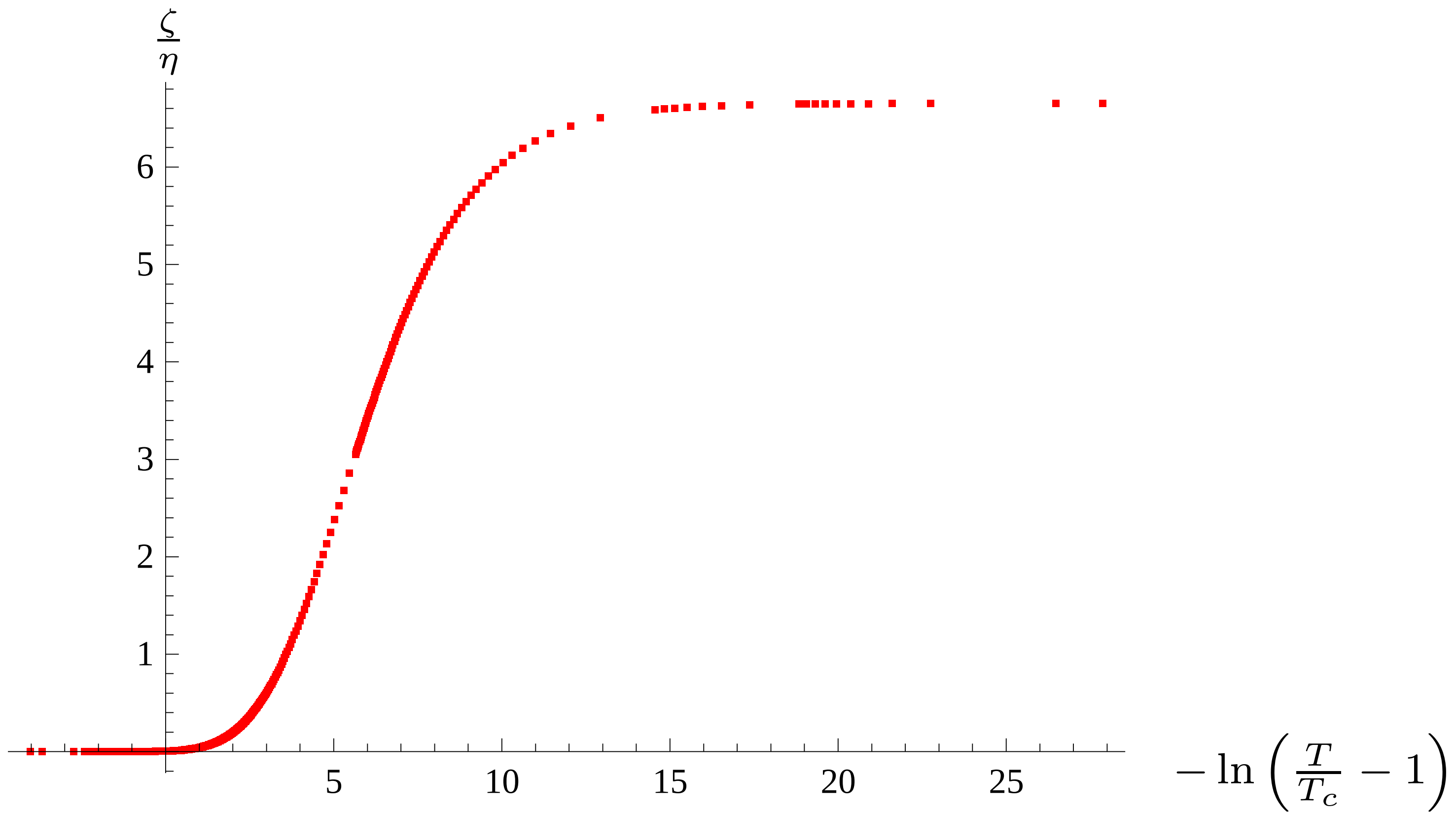


Figure 3: Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N}=2^*$ gauge theory plasma.

Note:

$$\frac{\zeta}{\eta}\Big|_{T=T_c}=6.65(3)\,,\qquad \text{or}\qquad \frac{\zeta}{s}\Big|_{T=T_c}=0.52(9)$$



which is close to the peak value $\frac{\zeta}{s}\sim 0.7$ extracted from lattice QCD by H.Myers.

The message so for:

- the RHIC collision occurs near the QCD phase transition
- the bulk viscosity can be rather substantial, and noticeably correct the conformal

hydrodynamics; in the holographic $\mathcal{N} = 2^*$ model at the phase transition

$$\eta \rightarrow \eta_{eff} \approx \eta \times 6$$

$\Rightarrow$ In their paper Song and Heinz take the idea of ’closeness’ to the phase transition seriously, and thus propose that one should expect as well ‘”critical slow down”, resulting in large relaxation time. This would lead to the strong memory effects (the sensitivity to the initial conditions). They run a bunch of hydrodynamic simulations to illustrate the point by playing with the value of the relaxation time.

**My question**: how does relaxation time occur in holographic models? Is it indeed enhanced near the phase transitions?

# Outline of the talk:

- Why should we care about the relaxation time
  - fundamental perspective
  - practical perspective
- Holographic bound on $\tau_{eff}$ in supergravity approximation
  - $\tau_{eff}$ in Kaluza-Klein reduction of a higher dimensional CFT hydrodynamics
  - $\tau_{eff}$ in the vicinity of a 4d fixed point deformed by relevant operators
  - $\tau_{eff}$ in $\mathcal{N} = 2^*$ plasma
  - $\tau_{eff}$ in phenomenological models of gauge/gravity correspondence
- $\tau_{eff}$ at weak coupling
- Beyond supergravity approximation for $\tau_{eff}$
  - finite coupling correction in $\mathcal{N} = 4$ plasma
  - effective relaxation time in Gauss-Bonnet plasma
- $\tau_{eff}$ near the phase transition
- Conclusions and further directions

**fundamental perspective:**

Causality constraints on the transport coefficients of the hydrodynamics

Hydrodynamics is an effective theory describing near-equilibrium phenomena in (relativistic) QFT:

$$\nabla_\nu T^{\mu\nu} = 0$$

The stress-energy tensor includes both an equilibrium part ($\mathcal{E}$ and $\mathcal{P}$ terms) and a dissipative part $\Pi^{\mu\nu}$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + \Pi^{\mu\nu} \,.$$

where $u^\mu$ is a local 4-velocity of the fluid and

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \,, \quad \Pi^\mu{}_\nu u^\nu = 0 \,, \quad u^\mu \nu_\mu = 0 \,,$$

Effective hydrodynamic description is equivalent to a derivative expansion of $\Pi^{\mu\nu}$ in local velocity gradients

Thus, to linear order in the derivative expansion

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu}(\eta, \zeta) = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}(\nabla_\alpha u^\alpha)$$

($\sigma^{\mu\nu} \propto \nabla_\nu u^\mu$) with $\{\eta, \zeta\}$ being the viscosity coefficients.

To simplify further discussion we consider only CFT's from now on: $\zeta = 0\,, \mathcal{E} = 3\mathcal{P}$. To second order in the derivative expansion

$$\Pi^{\mu\nu} = \Pi_1^{\mu\nu}(\eta) + \Pi_2^{\mu\nu}(\eta, \tau_\pi, \kappa, \lambda_1, \lambda_2, \lambda_3)$$

$$= -\eta\sigma^{\mu\nu} - \eta\tau_\pi \left[ \langle u \cdot \nabla\sigma^{\mu\nu} \rangle + \frac{1}{3}\,(\nabla \cdot u)\,\sigma^{\mu\nu} \right] + \mathrm{non-linear\,terms} + \cdots$$

$\Rightarrow$ It is straightforward to study dispersion relation of the linearized fluctuations in above theory

The dispersion relation of the shear channel fluctuations is given by

$$0 = -\mathfrak{w}^2 \, \tau_\pi T - \frac{i\mathfrak{w}}{2\pi} + \boldsymbol{k}^2 \, \frac{\eta}{s} \, ,$$

where $\mathfrak{w} = \omega/(2\pi T)$ and $\boldsymbol{k} = k/(2\pi T)$. Now the speed with which a wave-front propagates out from a discontinuity in any initial data is governed by

$$\lim_{|\boldsymbol{k}|\to\infty} \left. \frac{\mathrm{Re}(\mathfrak{w})}{\boldsymbol{k}} \right|_{[\mathrm{shear}]} = \sqrt{\frac{\eta}{s \, \tau_\Pi T}} \equiv v^{front}_{[\mathrm{shear}]} \, .$$

Hence causality in this channel imposes the restriction

$$\tau_\pi T \, \geq \frac{\eta}{s} \, .$$

**Notice:** the first-order hydrodynamics is recovered in the limit $\tau_\pi \to 0$, so causality is always violated at this order in the derivative truncation

Similar considerations in the sound channel imposes the (more stringent) restriction

$$\tau_\pi T \, \geq 2\frac{\eta}{s} \, .$$

So, the relaxation time is *required* to restore causality of relativistic effective theory of near-equilibrium dynamics, *i.e.,* the hydrodynamics.

$\Rightarrow$ One might worry that the causality constraint on the $\tau_\Pi$ is obtained from the regime outside the validity of the effective hydrodynamic approximation (derivative expansion is not valid in this regime). In general, the causality of the effective hydrodynamics depends on the microscopic parameters of the theory — in the CFT case, the central charges of the theory. In some models it can be shown what once the full non-equilibrium theory is causal, it's second-order truncated (in the velocity gradients) hydrodynamic description is causal as explained above.

**practical perspective:**

$\Rightarrow$ Even though first-order hydrodynamics is self-consistent in its regime of applicability, the numerical hydrodynamic simulations are typically unstable. Stability is restored with the introduction of the relaxation time. In other words: the breakdown of first-order hydro arises from the modes outside its regime of applicability but the computer does not know it!

## Holographic bound in $\tau_{eff}$ in supergravity approximation

$\Rightarrow$ How do we define the effective relaxation time?

The causal viscous relativistic hydrodynamics has many second order transport coefficients:

- in CFT cases - 5
- in non-CFT cases (see Romatschke) - 13

In practical simulations one usually introduces a single second-order transport coefficient (in order to limit the phenomenological parameter space). As a result, different simulations 'turn on' different combinations of the second order transport coefficients. In order to relate different hydrodynamic models, we introduce $\tau_{eff}$, defined from the sound wave dispersion relation as follows

$$\omega = \pm c_s k - i\Gamma k^2 \pm \frac{\Gamma}{c_s}\left(c_s^2 \tau_{eff} - \frac{\Gamma}{2}\right) k^3 + \mathcal{O}(k^4)\,,$$

where $c_s$ is the speed of the sound waves (obtained from the equation of state), and $\Gamma$ is the sound wave attenuation (determined by the shear and the bulk viscosities)

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}\,, \qquad \Gamma = \left(\frac{2}{3}\frac{\eta}{\mathcal{E}+\mathcal{P}} + \frac{1}{2}\frac{\zeta}{\mathcal{E}+\mathcal{P}}\right)\,.$$

As defined, $\tau_{eff}$ is

- the relaxation time of Müller-Israel-Stewart hydrodynamics
- it coincides with $\tau_\pi$
- in general non-conformal hydrodynamics of Romatschke

$$\tau_{eff} = \frac{\tau_\pi + \frac{3}{4}\frac{\zeta}{\eta}\,\tau_\Pi}{1 + \frac{3}{4}\frac{\zeta}{\eta}}\,.$$

**The Claim**:

$$\tau_{eff} T \;\geq\; \tau_\pi^{\mathcal{N}=4}\, T \;=\; \frac{2 - \ln 2}{2\pi} \equiv \tau_\pi^* T$$

In what follows I present various evidence for the above claim in the holographic setting, as well as for weak coupling

$\Rightarrow$ The simplest way to produce non-conformal hydrodynamics is to start with $d > 4$-dim conformal hydrodynamics, and reduce it to 4-dim on a flat $k$-torus.

- In $d$-dim CFT:

$$T^{\mu\nu} = \mathcal{E}^{(d)} u^\mu u^\nu + \mathcal{P}^{(d)} \triangle^{\mu\nu} + \Pi^{\mu\nu} \,, \qquad \triangle^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$\Pi^{\mu\nu} = -\eta^{(d)}\, \sigma^{\mu\nu} + \cdots$$

where

$$\sigma^{\mu\nu} = \triangle^{\mu\alpha}\triangle^{\nu\beta}\left[\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right] - \frac{2}{d-1}\, \triangle^{\mu\nu}\triangle^{\alpha\beta}\, \nabla_\alpha u_\beta$$

- To do the KK reduction of the CFT hydro, all we have to do is to assume

$$u^\mu = \left(v^i, \vec{0}\right)\,, \qquad i = 1 \cdots (d-k)$$

while keeping all the transport coefficients unchanged.

- The result now has to be interpreted in the framework of Romatschke hydrodynamics:

$$\Pi^{ij} = \pi^{ij} + \triangle^{ij}\Pi\,, \qquad \Pi = -\zeta(\nabla v) + \cdots$$

If

$$\{c_s^{(d)}, \Gamma^{(d)}, \tau_\pi^{(d)}\}$$

are the hydrodynamic coefficients of the $d$-dim CFT plasma, then

$$c_s = c_s^{(d)} = \frac{1}{\sqrt{d-1}}\,, \qquad \Gamma = \Gamma^{(d)}\,, \qquad \tau_{eff} = \tau_\pi^{(d)}$$

Using

$$\Gamma^{(d)} = \frac{d-2}{d-1}\,\frac{\eta^{(d)}}{\mathcal{E}^{(d)}+\mathcal{P}^{(d)}}$$

and the holographic universality for the shear viscosity

$$\frac{\eta}{s} = \frac{\eta^{(d)}}{s^{(d)}} = \frac{1}{4\pi}$$

we find

$$\frac{\zeta}{\eta} = 2\left(\frac{1}{3} - c_s^2\right)$$

Similar relaxations exist for the second order coefficients, for examples

$$\tau_\pi = \tau_\Pi = \tau_\pi^{(d)}$$

What is happening with the relaxation time bound?

Haack, Yarom, $\cdots$

$$\tau_\pi^{(d)} T = \frac{1}{4\pi}\left[\frac{d}{2} + H_{\frac{2}{d}}\right]$$

where

$$H_n = \int_0^1 dx\, \frac{1-x^n}{1-x}$$

is the harmonic number.

The relaxation time bound for these models is simply the statement that

$$\frac{1}{4\pi}\left[\frac{d}{2} + H_{\frac{2}{d}}\right] \geq \frac{2-\ln 2}{2\pi}\,, \qquad d \geq 4$$

$\Rightarrow \tau_{eff}$ in the vicinity of the fixed point

Consider an irrelevant deformation of a CFT in the UV:

$$\mathcal{L}_{CFT} \to \mathcal{L} = \mathcal{L}_{CFT} + \lambda_p \, \mathcal{O}^{(p)} \,, \qquad \dim \left[ \mathcal{O}^{(p)} \right] = p < 4$$

and $\lambda^{(p)}$ is a coupling. For large temperatures $T$,

$$T \gg (\lambda_p)^{1/(4-p)} \,,$$

above deformation is small, and can be treated perturbatively.

On the gravity side, above CFT deformation is dual to turning on a minimally coupled scalar of mass $m$ in the AdS background. If $L$ is the AdS radius,

$$p(4-p) = -(mL)^2$$

The non-normalizable mode of the scalar field (in appropriate normalization) should be set to be equal to $\lambda_p$.

Once the scalar is turned on, it will deform the AdS-Schwarzschild black brane solution, as well as the spectrum of its quasinormal mode. As explained by Kovtun and Starinets, the spectrum of a black brane quasinormal modes is equivalent to the hydrodynamic spectrum of the dual plasma. In particular, the lowest quasinormal mode of the graviton (with appropriate polarization) has to be identified with the sound wave in the plasma.

It is straightforward to compute such deformations, and extract the correction to $\tau_{eff}$. Instead of parameterizing the deformation in terms of $\lambda_p$, it is more convenient to parameterize it in terms of

$$\delta \equiv \frac{1}{3} - c_s^2$$

Note that

$$\left| \frac{(\lambda_p)^{1/(4-p)}}{T} \right| \ll 1 \qquad \Leftrightarrow \qquad |\delta| \ll 1$$

We find:

$$\tau_{eff} = \tau_\pi^\star \left( 1 + \beta_{[p]}\, \delta + \mathcal{O}(\delta^2) \right) ,$$

where $\tau_\pi^\star$ is the universal relaxation time of the holographic conformal hydrodynamics at (infinitely) strong coupling, *i.e.,* the lower bound, and $\beta_{[p]}$ is the correction induced by the operator with $\dim[\mathcal{O}] = p$. Explicitly we find,

$$\beta_{[p]} = \begin{cases} \text{2.2837(0)}\,, & p = 3\,, \\ \text{6.3016(8)}\,, & p = 2\,. \end{cases}$$

Again, in both cases the effective relaxation time bound is satisfied.

$\Rightarrow$ Effective relaxation of $\mathcal{N} = 2^*$ plasma.

It is possible to go beyond the leading order deformation of the CFT hydrodynamics in explicit examples of gauge theory/supergravity correspondence. In this way we find that

$$\tau_{eff}^{\mathcal{N}=2^*}\, T \;\geq\; \tau_\pi^* T$$

*i.e.,* the bound is satisfied.

$\Rightarrow$ $\tau_{eff}$ in phenomenological models of gauge/gravity correspondence

In arXiv:0902.2566 Todd Springer studied a phenomenological model of AdS gravity with a single scalar field. The scalar potential is chosen in such a way so that the hydro computations are simplified; in fact, is it possible to obtain (implicit) analytic expression for $\tau_{eff}$. Using the results of that paper, it is possible to verify to that the relaxation time bound is again satisfied.

## $\tau_{eff}$ at weak coupling

In the derivation of the second order viscous hydrodynamics from Boltzmann equations one finds that the effective relaxation time is (see Baier et.al, [arXiv:hep-ph/0602249])

$$\tau_{eff}^{Boltzmann} T = \frac{3\eta T}{2\mathcal{P}} = 6\,\frac{\eta}{s} \ \gtrsim\ \frac{3}{2\pi}\,,$$

which is much larger than $\tau_\pi^\star T$ since the ratio of the shear viscosity to the entropy density at weak coupling substantially exceed the KSS viscosity bound

## Beyond supergravity approximation for $\tau_{eff}$

$\Rightarrow$ Consider leading correction in $\frac{1}{\lambda}$ (the inverse 't Hooft coupling) in $\mathcal{N}=4$ SYM plasma. We find

$$\tau_{eff} T = \tau_\pi^* T + \frac{375}{32\pi}\zeta(3)\lambda^{-3/2} + \cdots$$

Notice that so far, the situation with the relaxation time bound is precisely parallel the status of the KSS viscosity bound, but **before** the Kats-Petrov paper!

$\Rightarrow$ We can do the analog of the Kat-Petrov computations, and ask how does the relaxation time is modified in CFT's with unequal central charges, *i.e.,* $c \neq a$. We find

$$\tau_{eff} T = \tau_\pi^* T - \frac{11}{16\pi}\,\frac{c-a}{a} + \mathcal{O}\left(\frac{(c-a)^2}{a^2}\right)$$

Since $(c-a)$ can be of either sign, effective relaxation time bound can be violated in a controllable string models.

Amusingly, the KSS bound can also be violated in non-supersymmetric CFT's; precisely when such violation occurs, the effective relaxation time bound is violated as well. (see M.Heller, R.Myers and AB)

## $\tau_{eff}$ near the phase transition

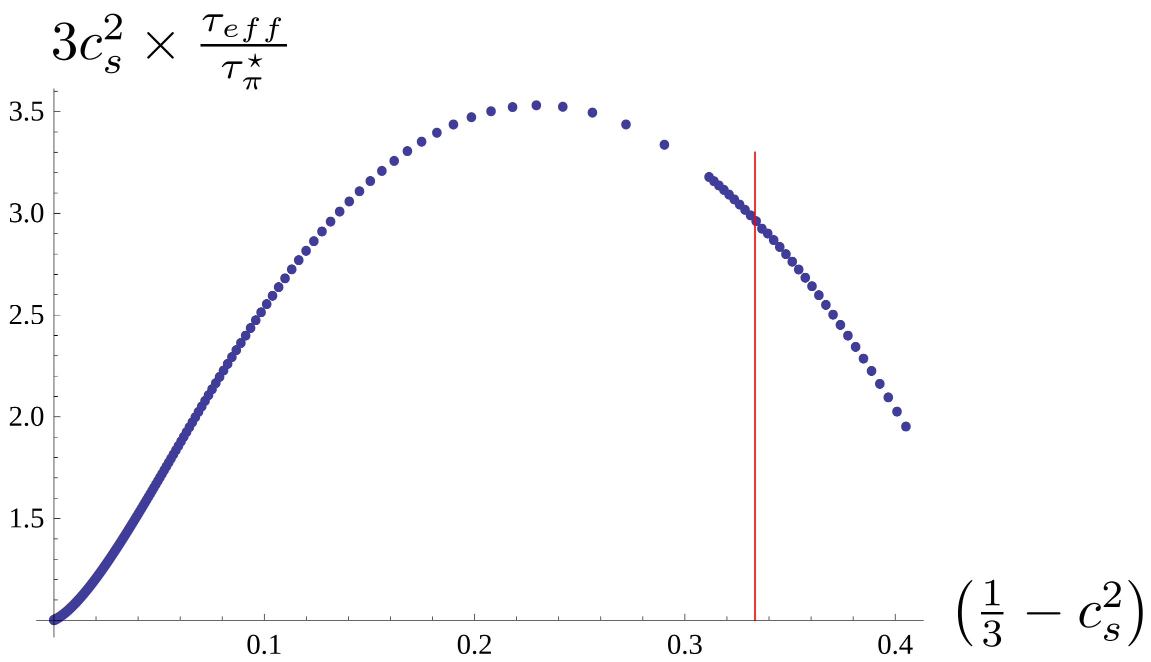


Figure 4: Effective relaxation time $\tau_{eff}$ of $\mathcal{N} = 2^*$ strongly coupled plasma. The vertical red line indicates a phase transition with vanishing speed of sound. Since $c_s^2 \propto (T - T_c)^{1/2}$ near the phase transition, $\tau_{eff} T_c \propto |1 - T_c/T|^{-1/2}$.

The critical slow-down suggested by Song and Heinz indeed happens in holographic models!

## Conclusions and further directions

- I introduced the notion of the effective relaxation time $\tau_{eff}$, and argued that it (or more precisely the dimensionless quantity $\tau_{eff}T$) is bounded at weak coupling and at infinite coupling in lots of models.
- The relaxation time bound is violated whenever the KSS viscosity bound is violated
- I showed that the effective relaxation time can diverge near the phase transitions in holographic models.

In the future:

- I'd like to better understand the link between the 'large' relaxation time and the sensitivity towards initial conditions. Some people question whether such relation indeed exist...
- How does the relaxation time behaves near the second-order phase transitions?