Finiteness, Complexity, and the Swampland

① Motivation

dream: existence of unifying principle underlying the (valid) swampland conjectures

maybe: "generalized finiteness principle"

entropy?, amplitudes?, information/complexity?

$\Longrightarrow$ very vague

Input from String theory? (standard approach)

arising effective theories are not generic but tied to geometry (even in 10d see F-theory)

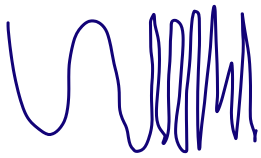
$\boxed{\text{Can we turn this into a principle?}}$

Example: Flux compactifications ($\rightarrow$ Graña)

$$V(z,\bar{z}) = \frac{1}{\mathcal{V}}\left(\int F_3 \wedge * F_3 + \dots\right)$$

very complicated function of $z, \bar{z}$,
but can be given geometrically
$\Rightarrow$ integrals over form on manifolds
"periods"

What could happen?

-  $\rightarrow \sin(1/x)$ infinitely many vacua

- min. valley:  $\rightsquigarrow$ infinite spirals : ↯ distance conjecture

But: this never happens!

Goals: ① Implement fundamentally that effective theories are geometric at core (without using string theory)

② and impose that each quantity has finite amount of information/complexity

$$\boxed{\text{finiteness of geometric complexity}}$$

How to make this precise?

Tool: "Swampland program" of mathematics:

Classification of mathem. theories/models

"Swampland":

e.g. Gödel's theorems $(\mathbb{Z}, \cdot, +)$ arithmetic

$\Rightarrow$ has undecidable statements

(e.g. solutions to certain Diophantine equations)

$\Rightarrow$ divide theory space by properties

Model theory (mathematical logic)

"geography of (tame) mathematics" (Hrushovski)

(developed in last century starting with Tarski)

$\Rightarrow$ Map of the model theoretic universe (google)

Here :

pick models that are

- o-minimal ⟹ $\boxed{\text{o-minimal structures}}$ van den Dries

⟹ answer to Grothendieck's dream of a topology for geometers

"tame topology"

basics : book by van den Dries "Tame topology and o-minimal structures"

- notion of complexity and information

⟹ $\boxed{\text{sharply o-minimal structures}}$

#o-minimal Binyamini, Novikov

② o-minimality as a finiteness principle

Definition: o-minimal structure

collection of subsets of $\mathbb{R}^n$ $(n \geq 0)$

"definable sets", "tame sets"

a) closed under

- finite unions, intersections, products, complements
- projections $\mathbb{R}^n \to \mathbb{R}^{n-1}$

b) contains zero-sets of all real polynomials

a)+b) $\Rightarrow$ "structure"

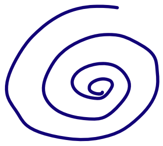
c) o-minimal: only definable subsets of $\mathbb{R}$ are finite unions of points and intervals ("$\leq$" definable)

- tame (definable) function: graph is tame set

$\Rightarrow$ $\boxed{\text{tame geometry:}}$ finite atlas of tame sets, tame coord. changes

Not o-minimal:

- integers, $\sin(x)$, $x \in \mathbb{R}$, $\sin(1/x)$ $x \in (0,1)$

 infinite spirals, ...

- certain non-oscillating fcts.

What is o-minimal?

- collection $\{ P_i(x_1 \dots x_n) = 0, \ \tilde{P}_j(x_1 \dots x_n) > 0 \}$

$\Longrightarrow$ $\mathbb{R}_{alg}$ "real alg. geom."

- specify which fcts. are fine $\mathcal{F} = \{ f_1, f_2, \dots \}$

gen. sets $\{ P_i(x_1 \dots x_n, f_1(x) \dots f_k(x)) = 0 \}$

- $\mathbb{R}_{exp}$, $\mathbb{R}_{an}$, $\mathbb{R}_{an,exp}$

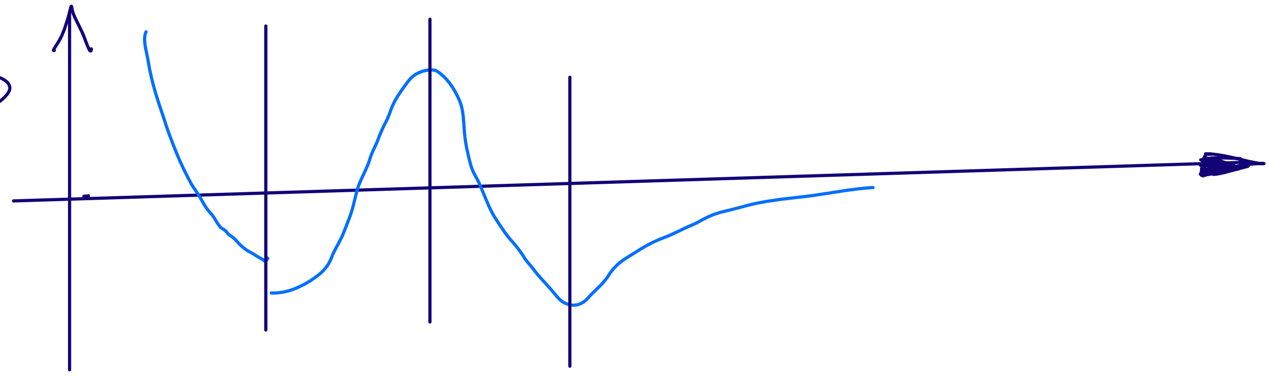
$\mathbb{R}_{an}$: restricted analytic e.g. $\sin(x)|_{[0,1]}$

- Pfaffian structure: $P(S)$ first order ✓ second order?

$$\frac{\partial f}{\partial x_i} = g_i(x, f) \qquad g_i \text{ in } S$$

- "exotic" structures: '22 structure defining $\Gamma(x)|_{(0,\infty)}$, $\zeta(x)|_{(1,\infty)}$

Basic theorem: $f: \mathbb{R} \to \mathbb{R}$ tame function

$\Longrightarrow$ 

- domain splits into finite number of open intervals on which f is
  - continious + monotonic
  - constant

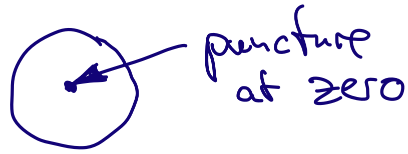
$\Longrightarrow$ max. finitely many jumps (discontinuities)

$\Longrightarrow$ finitely many minima, maxima + tame tail to infinity

Consequences:

- $f: \mathbb{C} \to \mathbb{C}$ holomorphic + tame $\Rightarrow$ polynomial
- $f: \Delta^* \to \mathbb{C}$ holomorphic + tame $\Rightarrow$ 0 is not essential sing.

punctured disk



- tameness can replace compactness (definable Chow)

"definable Chow" (Peterzil, Starchenko '08) more advanced: "tameness revolution"

proofs of longstanding mathematical conjectures: André-Oort

Ax-Schanuel (for Hodge Structures)

André-Grothendieck

new proof: Cattani-Deligne-Kaplan

famous theorem about "Hodge loci"

e.g. (2,2) fluxes on CY_4

BKT'20: period map / period integrals are tame functions in $\mathbb{R}_{an,exp}$

e.g.: $$W(z) = \int F_3 \wedge \Omega = \int_\gamma \Omega$$

$$= \# t^3 + \# t^2 + \dots \sum a_n e^{int}$$

large complex structure (but always true)

$W(z)$ is tame (definable in $\mathbb{R}_{an,exp}$)

③ Application to the flux lanscape

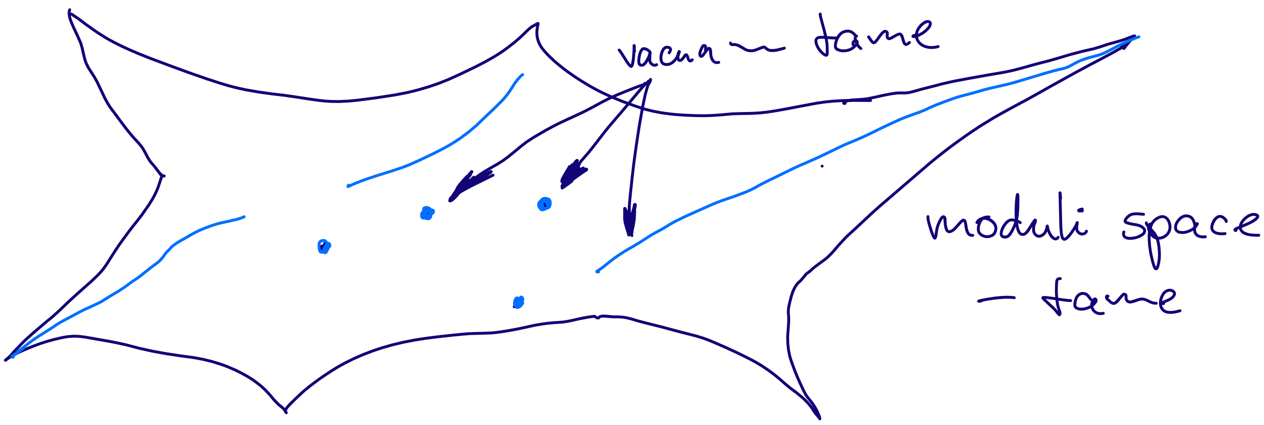
use general theorem of BGST '21 :

$\Longrightarrow$ shows that # of self-dual flux vacua with tadpole bound is finite

" vacum landscape is tame set "

$$*G_3 = i G_3 \quad \text{and} \quad \int_{Y_3} F_3 \wedge H_3 = K$$

or : $*G_4 = G_4$ and $\int_{Y_4} G_4 \wedge G_4 = K$



Finiteness statement follows from Swampland conjectures :

- Douglas , Acharya '03 : hep-th/0303194 '06 : hep-th/0606212

- "finiteness of effective theories valid below fixed cut-off" HMVV '21 2111.00015

10^{500} flux vacua ? $\Longrightarrow$ need numbers $\Rightarrow$ later

Exploring the landscape:

Hodge theory is tame:
(online lectures by Klingler, Tsimerman)

- flux compactification $\Rightarrow$ proofs of statements now possible
  tadpole conjecture?, ...

- $\partial_{z^i} W = 0$ and $W = 0$ has more eq. than unknowns: "unlikely intersection"

$\Rightarrow$ locus always comes with symmetry: CY is special

$\underbrace{\text{integer world}}_{\text{fluxes}}$ meets $\underbrace{\text{transcendental world}}_{\text{periods: } \int \Omega}$

- evidence for distance conj. from CY_3
  $\Rightarrow$ first results, but largely unexplored

- BPS states on CYs, BHs $\Rightarrow$ unexplored

- Feynman amplitudes are tame (DGS'22)
  but none of the none of the recent math breakthroughs use in this context

(4) Tameness conjecture: TG '21, DGS '23

$\boxed{\text{(a) space of effective theories (coupling fcts. field spaces, parameter spaces) that are valid below a } \underline{\text{fixed}} \text{ energy scale is tame} \quad \text{(b) physical observables are tame in parameters (compute on Euclidean + tame spacetime)}}$

<u>evidence</u>:

- implies previous finiteness conjectures (e.g. HMVV '21)
- <u>QFT</u>: finite-loop amplitudes in (renorm.) QFT are definable in $\mathbb{R}_{an,exp}$ (DGS '22)

tame QFT with cut-off Λ $\longrightarrow$ $\Lambda' < \Lambda$ tame QFT

finite vacua
$\downarrow$
finite vacua

- all String theory effective actions that I know are tame

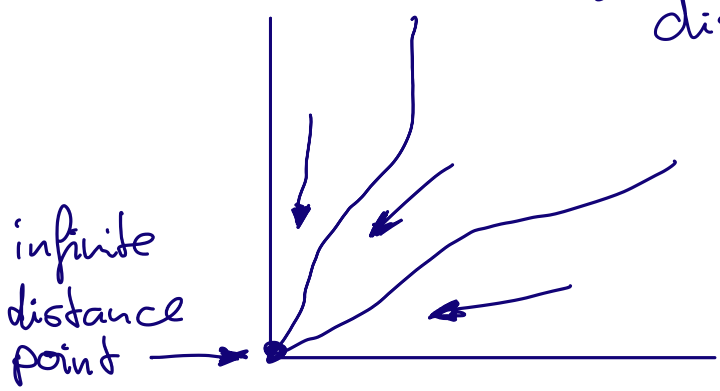
- relation to other swampland conjectures

  - no discrete infinite-order symmetries (as required for "no global symmetries in QG")

  - no potential with infinite spirals as min. (as required for distance conjecture)

  - finitly many sectors near infinite distance points ⇒ finitely many towers to satisfy distance conjecture

    (not along every path a new tower)



<u>but</u>: not extremly strong "no M_p"

⇒ need measure of information

⑤ Complexity and tameness

<u>recap</u>: introduced o-minimal structures
$\Rightarrow$ tame sets + functions

<u>next</u>: measure for "information content"
in these sets + fcts.

<u>example</u>: polynomial $a_1 x_1^3 x_2^1 + a_2 x_1^2 x_2^2 + \dots = 0$
number of free coeff. $\approx$ degree D
\# variables F

<u>QM</u>: $H = \frac{1}{2} p^2 + \frac{1}{2} g^2 x^2$

$$\Rightarrow \langle 0 | x(t_2) x(t_1) | 0 \rangle = \frac{2}{g} e^{-g(t_2 - t_1)}$$

$\Rightarrow$ need complexity of exponentia fct.

<u>Idea</u>: (Khovanskii, Gabrielov, Vorobjov)

ⓐ def. Pfaffian chain : $\varsigma_1 \dots \varsigma_r$

$$\frac{\partial \varsigma_i}{\partial x_j} = P_{ij}(x_1, \dots, x_n, \varsigma_1, \dots \varsigma_i)$$

$\rightarrow$ triangular!

ⓑ Pfaffian fct. $\quad f(x) = P(x_1, \dots, x_n, \varsigma_1, \dots, \varsigma_r)$

key point: P_{ij}, P are polynomials

$$\boxed{\begin{aligned} D &= \sum_{i,j} \mathrm{degr}(P_{ij}) + \mathrm{degr}(P) \\ F &= n + r \end{aligned}}$$

(F, D) "Pfaffian complexity"

What is meaning of (F, D) ?

topological: solutions to $f(x) = 0$

$$\# \leq \mathrm{Poly}(D)^{O(F)}$$

computational:

$$\mathrm{Comp} \leq \mathrm{Poly}(D)^{O(F)^{O(F)}} \equiv P_F(D)$$

(running time of algorithm to check statement)

very rougly in effective theory:

F – number of fields and non-trivial fcts.

D – how complicated are the functions

Examples:

- $f(x) = e^{ax}$ $\qquad \frac{\partial \varsigma}{\partial x} = a \varsigma$ $\qquad (F,D) = (2,2)$

  $\rightsquigarrow$ simple rule to def. exp.

- $f(x) = 1/x \Rightarrow (F,D) = (2,3)$

- $f(x) = a X^{2d} + b x^{d} \Longrightarrow (F,D) = (1, 2d)$

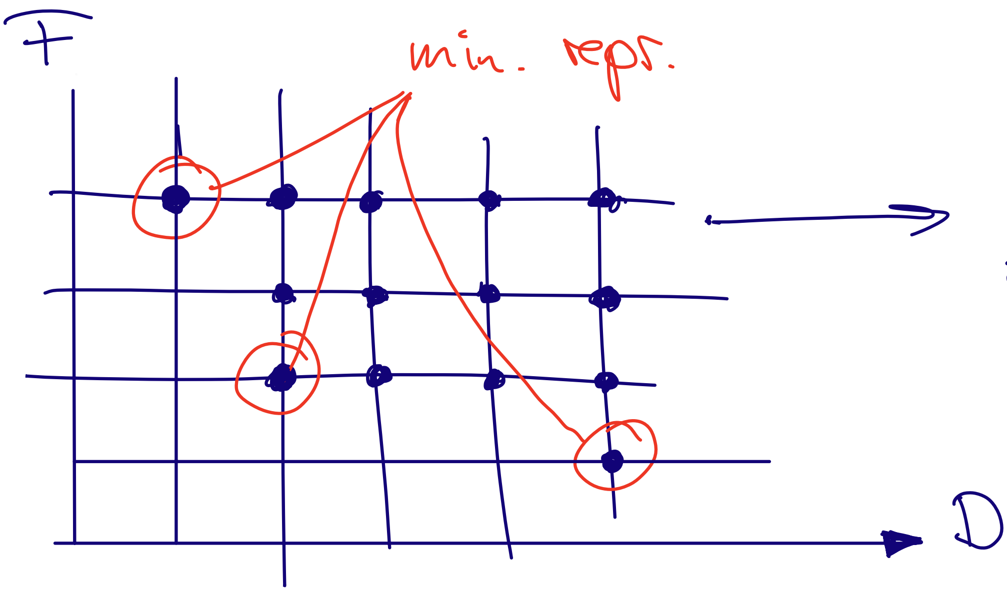
  but: "simpler" presentation?

  $\varsigma_1 = 1/x, \; \varsigma_2 = x^d$ $\qquad \frac{\partial \varsigma_2}{\partial x} = d \, \varsigma_1 \varsigma_2$

  $\Longrightarrow (F,D) = (3,6)$ $\qquad p_1 \sim 2, \; p_2 \sim 2$ $\qquad f(x^1) \sim 2$

up-shot: $\boxed{\text{complexity depends on representation of } f}$

define: # complexity $= \{(F,D) \text{ min. repr.}\}$



$\Longrightarrow$ can always repr. function in more complex way

- $f(x) = \cos(x)$ on $[-\pi, \pi]$

  $\varphi_1 = \tan(x/2)$ $\quad \varphi_2 = \cos^2(x/2)$ $\Rightarrow (F, D) = (3, 5)$

  <u>but</u>: $f_m(x) = \cos(mx)$ on $[-\pi, \pi]$

  $\Rightarrow (F, D) = (3, 4+m)$

<u>Fourier expansion:</u>

$$x^2 = \sum_{n=0}^{\infty} a_n \cos(nx) \approx \sum_{n=0}^{N} a_n \cos(nx)$$

simple

increasingly complex

strict $N \to \infty$ limit reduces complexity

<u>Recently</u> ('21, '22)

- Extension of construction to all sets of $\mathbb{R}_{\text{Pfaff}}$

  $\mathbb{R}_{\text{Pfaff}}$ ~ structure generated by all Pfaffian chains

<u>Note</u>: · challenge is to incorporate complement of set

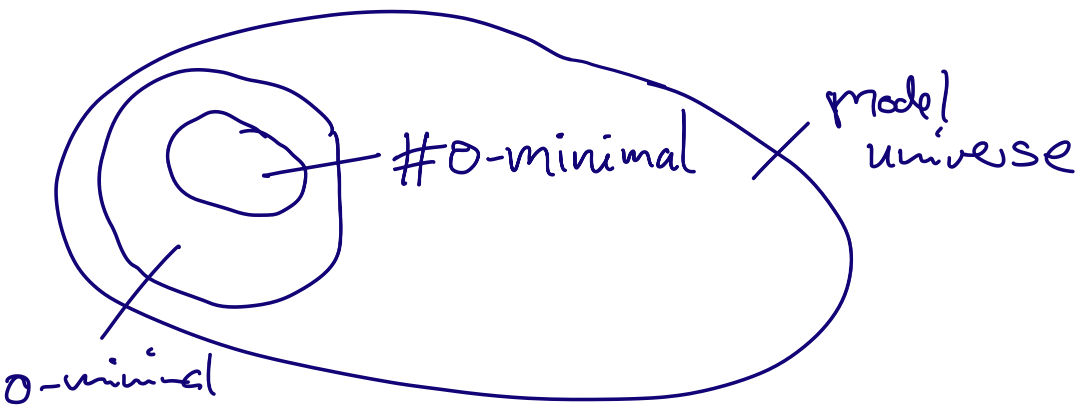
· explicit constr. for restr. Pfaffian fct.

Major advace:

- notion of sharply o-minimal structures: (#o-minimal) (see Binyamini, Novikov, Zack)

Every set/statement has (F, D) (finitely many min.)

> **Tameness axiom:**
>
> $\forall F \in \mathbb{N} \Rightarrow \exists$ computable $P_F(D)$ s.t. $A \subseteq \mathbb{R}$ has less than $P_F(D)$ connected comp
>
> polynom., positive coeff.



Note: gen. analytic fct. in $\mathbb{R}_{an}$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Longrightarrow \text{never sharply o-minimal}$$

infinitely many free parameters

Simple applications

a) <u>0-dim QFT</u>: $S = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \Rightarrow$ change: $\phi \to \sqrt{\frac{3}{2\lambda}}\, m\,\phi$

$$I_n = \int_{-\infty}^{\infty} d\phi \; \phi^n \; e^{-S} \qquad g = \frac{3m^4}{4\lambda}$$

$$Z(g) = I_0(g) = \sqrt{2}\; e^{g} K_{1/4}(g)$$

$\Rightarrow$ <u>note</u>: not analytic at $\lambda = 0 \Rightarrow$ transseries

mod. Bessel fct. of second kind

not in $\mathbb{R}_{an,exp}$

$\to$ resurgence

<u>trick</u>: transform second-order diff. eq. for I_0 to first order eqs. using

$$h(g) = -\frac{1}{I_0}\frac{\partial I_0}{\partial g}$$

$$\Rightarrow \quad (F,D)(Z) = (4,3)$$

$$(F,D)(I_n) = \left(4, 3 + \left\lceil \frac{n}{4} \right\rceil\right)$$

ceiling fct.

$\Rightarrow$ complexity D grows with # of insertion

<u>but</u> F const. due to <u>alg. relations</u>

b) <u>1-dim. QFT</u>: quantum mechanics (Euclidean time)

harmonic oscillator: $H = \frac{1}{2} p^2 + \frac{1}{2} g^2 x^2$

$$\langle 0 | x(t_2)\, x(t_1) | 0 \rangle = \frac{2}{g} e^{-g(t_2 - t_1)}$$

$$(F, D) = (3, 5)$$

<u>wavefunction?</u>

solve (time-indep.) Schrödinger eq.

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi}\right)^{1/4} \underbrace{e^{-\frac{m\omega x^2}{2}}}_{= S(x)} H_n\left(\sqrt{m\omega}\, x\right)$$

in variable x: $(F, D) = (2, 3+n)$

for $E_n = \left(n + \frac{1}{2}\right)\omega$

involved applications

$\boxed{\begin{array}{l}\underline{\text{Math conjecture}}\text{: Binyamini, Novikov '22} \\ \text{period integrals are } \#\text{ o-minimal}\end{array}}$

- perturbative amplitudes in (renorm.) QFTs
  $\Rightarrow$ use conjecture $\Rightarrow$ $(F, D)(\text{ampl})$?

- SU(N) Seiberg-Witten theory (4d $N=2$ Yang-Mills theory)
  $(F, D)\left(1/g_{SU(N)}\right)$?
  algebraic independence of periods
  $\Rightarrow$ $F \sim$ grows with N

Open problems:

- number of flux vacua $10^{500} - 10^{20000}$?
- compare with other notions of complexity
- Assign $(F, D)_{EFT}$ to effective theory
  $\Rightarrow$ compute for string theory examples
- What is behavior/bound on $(F, D)_{EFT}$?

References:

BGST '21 : 2112.06995

TG '21 : 2112.08383

GLL '22 : 2206.00697

DGS '22 : 2210.10057

DGS '23 : 2302.04275

GSV '23 : 2310.01484

many math refs. cited in these works