

## 1. Horizon regularity and surface gravity

Consider a static, spherically symmetric metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + g^2(r)d\Omega^2 \quad (1)$$

where  $f(r)$  vanishes at  $r = r_0$  linearly, and  $g(r_0) \neq 0$ .

Show that near  $r = r_0$  the metric is approximately

$$ds^2 \simeq -\kappa^2 \xi^2 dt^2 + d\xi^2 + g^2(r_0)d\Omega^2 \quad (2)$$

which is regular at  $\xi = 0$ , with

$$\kappa = \frac{f'(r_0)}{2}. \quad (3)$$

(assume that  $f'(r_0) > 0$ ).  $\kappa$  is called the surface gravity of the Killing horizon at  $r_0$ . Observe that we could rescale  $t \rightarrow \lambda t$  and  $\kappa$  would rescale too. Fixing this ambiguity is typically done by choosing a particular asymptotic static frame.

By performing a simple change of coordinates, find the form of the surface gravity for a metric of the form

$$ds^2 = -F(\bar{r})dt^2 + \frac{d\bar{r}^2}{G(\bar{r})} + \bar{r}^2 d\Omega^2. \quad (4)$$

Is this more or less general than the previous form? How must  $F(\bar{r})$  and  $G(\bar{r})$  behave near  $\bar{r}_0$  in order to approach the form (2)?

*Optional:*

Check that this result for  $\kappa$  is the same as the one obtained from the general formula for a Killing horizon generated by a vector  $k$

$$k^\mu \nabla_\mu k_\nu = \kappa k_\nu \quad (5)$$

(i.e., the non-afineness of the Killing generator of the horizon).

## 2. Mass, energy density, tension in $D$ dimensions

The mass of an object in GR is obtained by comparing its gravitational field at asymptotic infinity, where the field is weak, to the one created by a massive source in the linearized approximation to the Einstein equations.

Then, consider the Einstein equations in the linearized approximation around Minkowski space-time,  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ . Defining  $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$  and working in transverse gauge  $\nabla_\mu \bar{h}^{\mu\nu} = 0$  the linearized equations take the conventional form

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \quad (6)$$

Solve these equations for a pointlike mass

$$T_{tt} = M \delta^{(D-1)}(x^k). \quad (7)$$

You must have obtained a result for  $h_{tt}$  of the form

$$h_{tt} = \frac{\mu}{r^{D-3}} \quad (8)$$

where  $\mu$  is an  $M$ -dependent constant (the other  $h_{ij}$  are also non-zero but we will not use them). The numerical factors matter here: this result gives you a formula to compute the mass of any asymptotically flat solution from the asymptotic behavior of the metric in the region where the field is weak.

*Stress tensor of  $p$ -branes.* Now instead of a pointlike source, consider one that extends along  $p+1$  spacetime directions  $x^a$ ,  $a = 0, \dots, p$ , i.e., a  $p$ -brane,

$$T_{ab} = t_{ab} \delta^{(D-p-1)}(x^k), \quad (9)$$

where  $x^k$ ,  $k = p+1, \dots, D-1$  are the directions transverse to the brane. Solve the equations as before to find

$$h_{ab} = \frac{c_{ab}}{r^{D-p-3}} \quad (10)$$

(the other  $h_{ij}$  are also non-zero but we will not use them). Write  $t_{ab}$  as a function of  $c_{ab}$  to find

$$t_{ab} = \frac{\Omega_{(D-p-2)}}{16\pi G} [(D-p-3)c_{ab} + c\eta_{ab}] \quad (11)$$

where  $c = \eta^{ab}c_{ab}$  and  $\eta_{ab}$  is the Minkowski metric in  $p+1$  dimensions.

Finally, use this result to compute the energy density and pressure of a vacuum black brane built as the direct product of the Schwarzschild solution in  $D-p$  dimensions and a  $p$ -dimensional Euclidean space. Observe that even if  $c_{ii} = 0$ , the black string has non-zero tension.

*Optional: Newtonian dark stars in  $D$  dimensions*

In the Newtonian limit of General Relativity, the gravitational potential energy (energy per unit mass) is

$$\Phi = -\frac{1}{2}h_{tt}. \quad (12)$$

Follow John Michell and Simon de Laplace and compute, using Newtonian mechanics, the radius  $r_{ML}$  at which the escape velocity of a test particle in this field reaches the speed of light. This is the radius of a Newtonian ‘dark star’ in higher dimensions. Check that this is exactly the same, in any  $D$ , as the horizon radius of the higher-dimensional Schwarzschild black hole.

### 3. Boosted black strings

A boosted black string and a static one differ only by a diffeomorphism (namely, a Lorentz boost). Is it then appropriate to regard them as different solutions? More precisely, under what circumstances are they indeed different solutions? If we take the (extremal) limit of boost velocity to speed of light, is the metric still diffeomorphic to the static black string?

### 4. Quantization of momentum

Use (11) to compute the energy and momentum densities,  $t_{tt}$ ,  $t_{tx}$  of a black string boosted to the speed of light

$$ds^2 = -dt^2 + dx^2 + \frac{q}{r^n}(dt + dx)^2 + dr^2 + r^2 d\Omega_{(n+1)}^2. \quad (13)$$

Should you be surprised that they are equal?

If the string is compactified,  $x \sim x + 2\pi R$ , then in the quantum theory the total momentum must be quantized  $\int dx t_{tx} = N/R$ ,  $N \in \mathbf{Z}$ . Find how the momentum number  $N$  is related to the charge  $q$ . For a ten-dimensional string, use that  $16\pi G = (2\pi\ell_s)^7 g^2 \ell_s$  to express the relation between  $q$  and  $N$  in terms of the string coupling  $g$  and string length  $\ell_s$ .

1. Surface gravity

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + g^2(r) d\Omega^2$$

$$f(r) = f'(r_0) (r-r_0) + O(r-r_0)^2$$

$$ds^2 \approx -f'_0 (r-r_0) dt^2 + \frac{dr^2}{f'_0 (r-r_0)} + g^2(r_0) d\Omega^2$$

$$f'_0 \equiv f'(r_0)$$

Proper radius:  $\frac{dr}{\sqrt{f'_0 (r-r_0)}} = d\xi \Rightarrow \xi = 2 \sqrt{\frac{r-r_0}{f'_0}} \quad r-r_0 = \frac{f'_0}{4} \xi^2$

$$\Rightarrow ds^2 \approx -\kappa^2 \xi^2 dt^2 + d\xi^2 + g_0^2 d\Omega^2 \quad \text{w/ } \kappa = \frac{f'_0}{2}$$

$ds^2 = -F(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2$  ; This is "area-radius gauge, just as general as the previous one,  $r \rightarrow g(r)$

Either:  $F(r), g(r)$  have simple zeroes at  $r=r_0$   
 or  $F(r)$  has a double zero, at  $r=r_0$ , &  $g(r_0) \neq 0$

$$\kappa = \frac{1}{2} \sqrt{\frac{g_0}{F_0}} F'_0 \quad \left\{ \begin{array}{l} \frac{1}{2} \sqrt{g'_0 F_0} \\ \sqrt{\frac{g_0 F_0''}{2}} \end{array} \right.$$

L'Hôpital

$$\frac{dr}{\sqrt{f}} = \frac{d\bar{r}}{\sqrt{g}}$$

$$f(r) = F(\bar{r})$$

Chain rule

$$\frac{df}{dr} = \frac{dF}{d\bar{r}} \frac{d\bar{r}}{dr} = F' \sqrt{\frac{g}{F}}$$

$$\Rightarrow f'_0 = F'_0 \sqrt{\frac{g_0}{F_0}}$$

2. The main point is the correct normalization in the solution to the distributional Poisson eq

$$\nabla^2 \phi = \delta^{(n+2)}(x) \quad \text{in } n+2 \text{ dimensions} \quad dr^2 + r^2 d\Omega_{(n+1)}^2$$

Take  $\phi = \frac{B}{r^n}$       $\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu = \frac{1}{r^{n+1}} \partial_r r^{n+1} \partial_r$

Integrate over a ball  $r \leq R$  with boundary  $S^{n+1}$  of radius  $R$

$$\int_{\text{ball}} \nabla^2 \phi = 1$$

$$\int_{S^{n+1}} n \cdot \nabla \phi = \Omega_{n+1} R^{n+1} \partial_r \phi \Big|_{r=R} = -n \Omega_{n+1} B$$

$$\Rightarrow B = -\frac{1}{n \Omega_{n+1}}$$

ie  $\nabla^2 \frac{1}{r^n} = -n \Omega_{n+1} \delta^{(n+2)}(r)$

Thus  $\square \bar{h}_{tt} = -16\pi G M \delta^{(D-1)}(x) \quad n = D-3$

$$\Rightarrow \bar{h}_{tt} = \frac{16\pi G M}{(D-3) \Omega_{D-2}} \frac{1}{r^{D-3}}$$

$$\bar{h} = \gamma^{\mu\nu} \bar{h}_{\mu\nu} = h - \frac{D}{2} h = \frac{2-D}{2} h \quad \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{D-2} \gamma_{\mu\nu} \bar{h}$$

Here  $\bar{h} = -\bar{h}_{tt}$

$$h_{tt} = \bar{h}_{tt} - \frac{1}{D-2} \bar{h}_{tt} = \frac{D-3}{D-2} \bar{h}_{tt} = \frac{16\pi G M}{(D-2) \Omega_{D-2}} \frac{1}{r^{D-3}}$$

• Mitchell-Laplace

Potential  $\Phi = -\frac{1}{2} h_{tt}$

Escape:  $\Phi + K = 0 \Rightarrow h_{tt} = 1 \Rightarrow r_{HL}^{D-3} = \frac{16\pi G M}{(D-2) \Omega_{D-2}} = r_{Schw}^{D-3}$

Kinetic  $K = \frac{1}{2} \dot{c}^2 = \frac{1}{2}$

• Brane:  $t_{ab} = \frac{\Omega_{n+1}}{16\pi G} (n C_{ab} + C \eta_{ab}) \quad C = \gamma^{ab} C_{ab} \quad n = D-p-3$

Perfect fluid

$$\epsilon = \frac{\Omega_{n+1}}{16\pi G} [(n+1) C_{tt} - \overset{\text{no sum in } i}{\downarrow} p C_{ii}] \quad P = \frac{\Omega_{n+1}}{16\pi G} [(D-3) C_{ii} - C_{tt}]$$

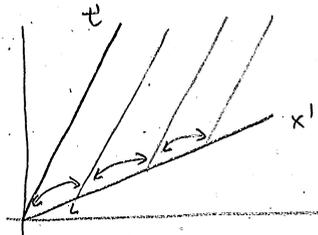
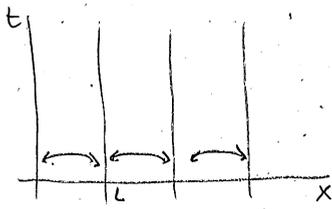
Black brane:  $C_{tt} = r_0^n \quad C_{xx} = 0 \quad C = -r_0^n$

$$\epsilon = \frac{\Omega_{n+1}}{16\pi G} (n+1) r_0^n$$

$$P = -\frac{1}{n+1} \epsilon$$

3.- Hint: compactification of string direction  $x \sim x + 2\pi R$  breaks Lorentz invariance  
so then solutions are not diffeomorphic

It is different to identify  $x \sim x + L$  at constant  $t$   
than to identify  $x' \sim x' + L$  at constant  $t'$



with  
 $x' = \cosh \alpha x + \sinh \alpha t$   
 $t' = \sinh \alpha t + \cosh \alpha x$

Light-like limit of boost is singular: it can't be inverted

$$4.- \quad ds^2 = -dt^2 + dx^2 + \frac{q}{r^n} (dt+dx)^2 + dr^2 + r^2 d\Omega_{(n+1)}^2$$

$$n = D-4$$

$$h_{tt} = \frac{q}{r^n} \quad c_{tt} = c_{xx} = c_{tx} = q$$

$$h_{xx} = \frac{q}{r^n} \quad c = 0$$

$$h_{tx} = \frac{q}{r^n} \quad t_{ab} = \frac{\Omega_{n+1}}{16\pi G} n c_{ab}$$

$$\Rightarrow t_{tt} = t_{xx}$$

$$M = \int dx t_{tt}$$

$M = P$  ; massless particle

$$P = \int dx t_{tx}$$

$$P = 2\pi R \frac{\Omega_{n+1}}{16\pi G} n q = \frac{N}{R}$$

$$\boxed{q = \frac{8G}{(D-4)\Omega_{D-3}} \frac{N}{R^2}}$$

$$D=10 \quad \Omega_7 = \frac{\pi^4}{3}$$

$$q = \frac{(2\pi l_s)^6}{6\Omega_7} \left(\frac{l_s}{R}\right)^2 g^2 N$$

Upon T-duality on x

## 1. Black Hole thermodynamics from Euclidean Quantum Gravity

In the semiclassical approximation to the Euclidean Quantum Gravity Path Integral, the thermodynamical partition function and free energy are given by

$$Z[\beta] = e^{-\beta F} \approx e^{-I_E[g^{(\text{cl})}]} \quad (1)$$

where  $I_E[g^{(\text{cl})}]$  is the Euclidean action of a classical solution  $g^{(\text{cl})}$  with periodic boundary conditions in imaginary time,  $\tau \sim \tau + \beta$ . For an asymptotically flat geometry, such as the Schwarzschild solution, this action is actually infinite and needs renormalization. A conventional way to do it is by subtracting the action  $I_E[g^0]$  of a reference background spacetime  $g^0$ . In this case the natural background to take is the Minkowski vacuum, which can be regarded as the ground state for asymptotically flat boundary conditions. If we are dealing with vacuum solutions,  $R_{\mu\nu} = 0$ , the bulk term (Einstein-Hilbert) in the action vanishes and the calculation reduces to the boundary term

$$I_E[g^{(\text{cl})}] - I_E[g^0] = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{h}(K - K^0), \quad (2)$$

where the integral is taken at the asymptotic boundary of spacetime, and one must make sure that at the boundary  $\partial\mathcal{M}$  the geometries of the black hole spacetime and of the Minkowski background are isometric. We shall use this method to compute the free energy of the Schwarzschild black hole.

- To regularize the calculation, consider the boundary to be at a large but finite constant radius  $r = R$ . Eventually the cutoff  $R$  will be removed by sending  $R \rightarrow \infty$ . The boundary metrics are

$$\begin{aligned} ds^2(\partial\mathcal{M}) &= h_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{R}\right) d\tau^2 + R^2 d\Omega_2, \\ ds_0^2(\partial\mathcal{M}) &= h_{\mu\nu}^0 dx^\mu dx^\nu = d\tau_0^2 + R^2 d\Omega_2. \end{aligned} \quad (3)$$

Since the metrics on the angular part already coincide, we see that in order for the two boundary metrics to be the same we only have to match the length of the circles of Euclidean time,

$$\int_0^\beta d\tau \sqrt{h_{\tau\tau}} = \int_0^{\beta_0} d\tau_0 \sqrt{h_{\tau\tau}^0}. \quad (4)$$

This fixes the value of the  $\tau$ -periodicity  $\beta_0$  of the background, for a given value of  $\beta$  for the black hole.

- Next, compute the integrands in (2), using  $\sqrt{h}K = n^\mu \partial_\mu \sqrt{h}$ , where  $n$  is the outward radial unit normal to the boundary. After performing the integrals in (2), the limit  $R \rightarrow \infty$  should yield a finite result. The free energy, relative to the Minkowski ground state, is  $I_E[g^{(\text{cl})}] - I_E[g^0] = \beta F$ .

- Expressing  $F$  as a function of the temperature, use the conventional thermodynamics expressions

$$E = \frac{\partial(\beta F)}{\partial\beta}, \quad S = \left(\beta \frac{\partial}{\partial\beta} - 1\right)(\beta F), \quad (5)$$

to obtain the energy and entropy. Check that these agree with the expected results.

- (*Optional*). If you have enjoyed this exercise, then you may want to try your hand at AdS-Schwarzschild black holes in  $d$  dimensions, which (in the Euclidean section) are

$$ds^2 = \left(1 - \frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{\mu}{r^{d-3}} + \frac{r^2}{\ell^2}} + r^2 d\Omega_{(d-2)}, \quad (6)$$

and are solutions of the equations of the Euclidean action

$$I = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \sqrt{g} \left( R + \frac{(d-1)(d-2)}{\ell^2} \right) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{h} K. \quad (7)$$

Taking AdS space ( $\mu = 0$ ) as the background for subtraction, the calculation is only a little more involved than the previous one. Hints: (i) the bulk Einstein-Hilbert term in the action does not vanish on-shell now; (ii) for calculations, it is more practical to use the horizon size  $r_+$  as the parameter in the solution instead of  $\mu$ .

When you are done, observe that you have obtained an expression for the energy, *i.e.*, mass of the AdS black hole in  $d$  dimensions. If you take the limit  $\ell \rightarrow \infty$  this should coincide, including all factors, with the expression for the  $d$ -dimensional mass as a function of  $\mu$  that you obtained in a previous exercise.

## 2. Entropy of astrophysical black holes (*Optional, for fun*)

The Bekenstein-Hawking entropy

$$S_{BH} = \frac{c^3}{\hbar G} \frac{\mathcal{A}_H}{4} \quad (8)$$

is enormous for astrophysical black holes. Here we have taken Boltzmann's constant  $k_B = 1$ , so temperature is measured in units of energy and the entropy is dimensionless. Ignoring rotation (which would only introduce corrections by factors of order one) we can write

$$S_{BH} \simeq 10^{77} \left( \frac{M}{M_\odot} \right)^2, \quad (9)$$

where  $M_\odot = 2 \times 10^{33}$  g is the mass of the Sun. In the following, we will try to obtain order-of-magnitude estimates of the entropy of astrophysical black holes and compare it to the entropies of other relevant systems. Our assumptions will be rather crude and may be off by one or even two orders of magnitude, but the comparisons will still be significant.

• **Entropy of a star.** A very crude estimate (but sufficient for our purposes) of the entropy of the Sun is the following. For an ideal gas of  $n$  particles, the entropy is  $S \sim n$ . Regard the Sun as a ball of a gas of particles of mass equal to the proton mass,  $m_p \sim 10^{-24}$ g. What is then the entropy of the Sun? What is the entropy of a black hole of the same mass?

• **Entropy of a galaxy.** Estimate in the same manner the entropy of the galaxy from the following sources:

1. Luminous matter (stars), if the luminous mass in the galaxy is  $M_{\text{galaxy}} \sim 10^{11} M_\odot$ .
2. Central black hole. There is significant evidence that our galaxy contains a central black hole with mass  $M_\bullet \simeq 3 \times 10^6 M_\odot$ .
3. Stellar-mass black holes. The number of black holes of stellar mass  $\sim M_\odot$  in the galaxy is estimated to be  $\sim 10^8$ .

Which of these three contributions to the entropy of the galaxy is the largest?

• **Entropy of the Universe.** Estimate the following contributions to the entropy of the Universe:

1. Luminous matter (stars in galaxies): take the radius of the Universe to be  $\sim 10^{10}$  ly (ly = light year), and consider that each galaxy occupies a sphere of radius  $\sim 10^5 - 10^6$  ly, with our galaxy being a typical galaxy.
2. Cosmic microwave background radiation at  $T \sim 2\text{K} \sim 10^{-4}\text{eV}$ . The entropy of a gas of photons at temperature  $T$  in a volume  $V$  is  $S \sim VT^3$  (with  $\hbar = 1 = c$ ). (*Hint*: write  $T$  in terms of the wavelength of the radiation). Incidentally, this entropy is essentially the number of photons of the CMB.
3. Entropy in black holes at galactic centers. Assume that each galaxy has a central black hole of mass  $\sim 10^6 - 10^9 M_\odot$ .

# 1.- BH euclidean Thermo

No. \_\_\_\_\_

Date . . . . .

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$$

$$ds_0^2 = dt_0^2 + dr^2 + r^2 d\Omega^2$$

$$r = R \gg GM$$

Match Time circles  $\beta \sqrt{1 - \frac{2GM}{r}} = \beta_0$

Normal:  $n^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{1 - \frac{2GM}{r}}$

$$\sqrt{h} = \sqrt{1 - \frac{2GM}{r}} r^2 \sin\theta$$

$$\sqrt{h} K = \sqrt{1 - \frac{2GM}{r}} \partial_r \left( \sqrt{1 - \frac{2GM}{r}} r^2 \sin\theta \right) \Big|_{r=R} \approx \sin\theta \left(1 - \frac{GM}{R}\right) (2R - GM) \approx 2R \sin\theta \left(1 - \frac{3}{2} \frac{GM}{R}\right)$$

$$\sqrt{h} K^0 = 2R \sin\theta$$

(expand in  $\frac{GM}{R} \ll 1$ )

$$\int \sqrt{h} (K - K^0) = \int \underbrace{d\theta d\phi \sin\theta}_{4\pi} \left[ \int dt 2R \left(1 - \frac{3}{2} \frac{GM}{R}\right) - \int dt_0 2R \right]$$

$$= \beta 8\pi R \left(1 - \frac{3}{2} \frac{GM}{R}\right) - \beta_0 8\pi R \approx -4\pi \beta GM$$

$\hookrightarrow \sim \beta \left(1 - \frac{GM}{R}\right)$

$$\Rightarrow \beta F = -\frac{1}{8\pi G} \int \sqrt{h} (K - K^0) = \frac{1}{2} \beta M$$

$$E = \frac{\beta}{8\pi G} = M$$

$$S = \frac{\beta^2}{16\pi G} = \frac{A_H}{4G}$$

## 2 - Entropy of astrophysical black holes

$$S_{BH} = \frac{c^3}{4Gk_B} \frac{A_H}{4}$$

$k_B = 1$ : dimensionless entropy

$$\sim 10^{76} \left(\frac{M}{M_\odot}\right)^2$$

$$M_\odot = 2 \times 10^{33} \text{ g}$$

$$M_{pe} \sim 2 \times 10^{-5} \text{ g}$$

(This can be easily found as  $S_{BH} \sim \left(\frac{M}{M_{pe}}\right)^2 = \left(\frac{M}{M_\odot}\right)^2 \left(\frac{M_\odot}{M_{pe}}\right)^2 \sim \left(\frac{M}{M_\odot}\right)^2 \left(\frac{2 \times 10^{33} \text{ g}}{2 \times 10^{-5} \text{ g}}\right)^2 \sim 10^{76} \left(\frac{M}{M_\odot}\right)^2$ )

Entropy of a star: for an ideal gas of  $n$  particles,  $S \sim n$

if made up of nucleons,  $m_p \sim 10^{-24} \text{ g}$

$$\Rightarrow S \sim n \sim \frac{10^{33}}{10^{-24}} \sim 10^{57}$$

ie  $S_{BH}(M_\odot) \gg S_\odot$

Entropy of galaxy (luminous matter):  $M_{gal} \sim 10^{11} M_\odot$

$$\Rightarrow S_{gal} \sim 10^{69} \ll S_{BH}(M_\odot)$$

Entropy of central bh:  $M_{BH} \sim 3 \times 10^6 M_\odot$   $S_{BH, \text{center}} \sim 10^{76} (3 \times 10^6)^2 \sim 10^{89}$

" " stellar-mass bh in galaxy:  $\# \sim 10^8$   $S_{BHs} \sim 10^8 10^{76} \sim 10^{84} \ll S_{\text{central BH}}$

The central bh dominates the galactic entropy

Universe: (i) luminous matter

$$\left. \begin{array}{l} R_{univ} \sim 10^{10} \text{ ly} \\ R_{gal} \sim 10^{22} \sim 10^{26} \text{ ly} \end{array} \right\} N_{gal} \sim \left(\frac{10^{10}}{10^{22-26}}\right)^3 \sim 10^{12-15}$$

$$S_{lum} \sim 10^{12-15} S_{gal} \sim 10^{80-83}$$

(ii) CMBR:  $S \sim VT^3$   $10^{10} \text{ ly} \sim 10^{28} \text{ cm}$

$$V \sim (10^{28} \text{ cm})^3 \sim 10^{84} \text{ cm}^3$$

$$k_B \sim 3 \times 10^{-16} \text{ K cm}^{-1}$$

$$3K \sim 1 \text{ min}^{-1}$$

$$T \sim 3K \sim \frac{1}{0.1 \text{ cm}} = 10 \text{ cm}^{-1}$$

$$S_{CMBR} \sim VT^3 \sim 10^{84} \text{ cm}^3 \cdot 10^3 \text{ cm}^{-3} \sim 10^{87} \sim N_{\text{photons}}$$

(iii) BHs (galactic centers)  $M_\odot \sim 10^{6-9} M_\odot$

$$S_{1BH} \sim 10^{77} (10^{6-9})^2 \sim 10^{89-95}$$

$$S_{BH \text{ tot}} \sim N_{gal} S_{1BH} \sim 10^{100} \sim 10^{110} \gg S_{CMBR} > S_{lum}$$

$$N_{gal} \sim 10^{12-15}$$