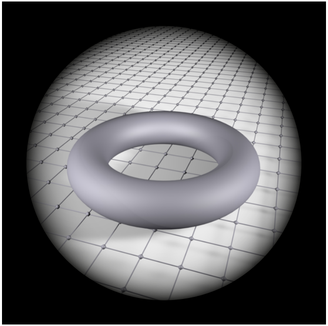
Creating and Probing Topological Matter with Cold Atoms: From Shaken Lattices to Synthetic Dimensions

Nathan Goldman



2015 Arnold Sommerfeld School, August–September 2015

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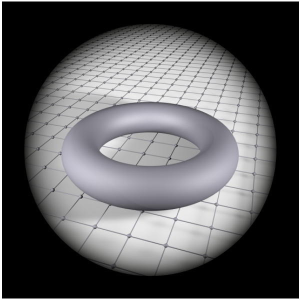
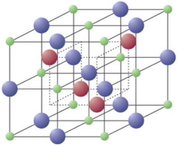
General introduction:
From “real” materials to cold atoms



Quantum Simulation with cold atoms: From real materials to optical lattices

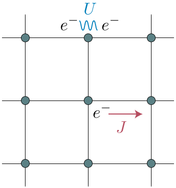
Real materials



Some examples:

- Superconductors
- Graphene
- Topological insulators
- Weyl semimetals

Theoretical models

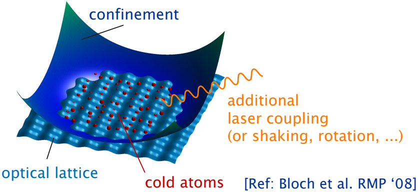


$$\hat{H}_{\rm model} = -J \sum_{\rm link} \hat{a}_j^\dagger \hat{a}_{j+1}$$

$$+ U \sum_{\rm sites} \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j$$

$$+ \text{ ingredients } \dots$$

Cold atoms in optical lattices



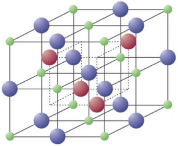
[Ref: Bloch et al. RMP '08]

- $\hat{H}_{\rm model} = \hat{H}_{\rm atom}$
- Control **lattice geometry** through light-field intensity

  $V(\boldsymbol{r}) = \frac{1}{2}\alpha(\lambda)|\boldsymbol{E}(\boldsymbol{r})|^2$: optical dipole potential
- Control over microscopic parameters: U , J , ...
- Clean: no impurity, no phonons,...
- Load Fermi gases, or Bose gases, or mixtures...

Quantum Simulation with cold atoms: From real materials to optical lattices

Real materials



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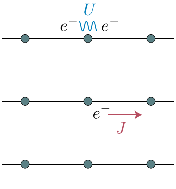

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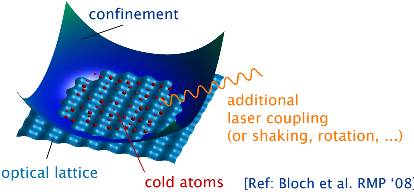
Cold atoms in optical lattices



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Some examples:

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- Graphene
- Topological insulators
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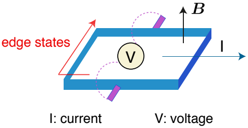
A few goals and challenges...

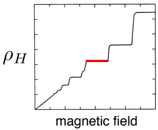
- **Identify** cold-atom setups that simulate systems of interest (real materials, but also high-energy physics?)
- **Detect** the effects using available probes (imaging techniques, single-site addressing, spectroscopy,...)
- **Go beyond** solid-state physics: "observe things that can't be created or seen in solids", identify new effects, ...

Our main interest in these lectures: topological states of matter

- **The quantum Hall effect**

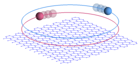


$$\sigma_H = 1/\rho_H = \nu \times (e^2/h)$$



strong magnetic field!

- **2D topological insulators** (quantum spin Hall effect)



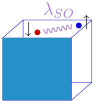
Helical edge states

$$\boldsymbol{j}_{\text{spin}} = \boldsymbol{j}_{\uparrow} - \boldsymbol{j}_{\downarrow}$$

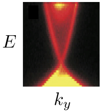
strong spin–orbit coupling!

$$\hat{H}_{\text{SO}} = \sum_{\mu,\nu} \alpha_{\mu,\nu}\, k_\mu \hat{\sigma}_\nu$$

- **3D topological insulators** (Dirac–fermion surface states, axion electrodyn.)



Helical Dirac fermion
(on the surface)

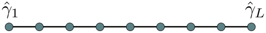


$$E \sim vk$$

Xia et al. Nat. Phys. 2009

strong spin–orbit coupling!

- **Topological superconducting wires** (Majorana fermions at the edges)



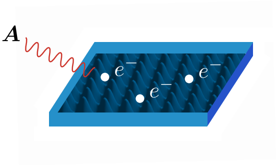
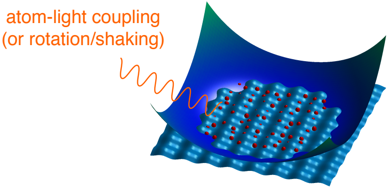
$$\hat{\gamma} = \hat{\gamma}^\dagger$$

$$\hat{\psi}_{\text{edge}} = (\hat{\gamma}_1 + i\hat{\gamma}_L)/2$$

spin–orbit coupling + s–wave supercond.

[Refs: Hasan & Kane RMP ‘10, Qi & Zhang RMP ’11]

Synthetic gauge potentials: a route towards topological atomic states

Electrons in a solid	**Neutral atoms** in optical lattices
	
$\hat{H} = \sum_j \frac{1}{2m}\left(\boldsymbol{p}_j - q\boldsymbol{A}(\boldsymbol{x}_j)\right)^2 + V(\boldsymbol{x}_j) + \ldots$	$\hat{H} = \sum_j \frac{1}{2m}\left(\boldsymbol{p}_j - \kappa\boldsymbol{A}(\boldsymbol{x}_j)\right)^2 + V(\boldsymbol{x}_j) + \ldots$
q : the electron charge $\boldsymbol{A}$: gauge potential	κ : coupling constant $\boldsymbol{A}$: synthetic gauge potential

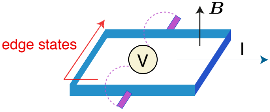
Ex: Magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ $\longrightarrow$ synthetic magnetic field for neutral atoms

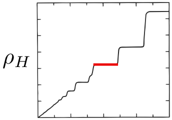
Spin-orbit coupling $A_\mu \sim \hat{\sigma}_{x,y,z} \in \mathfrak{su}(2)$ $\longrightarrow$ synthetic spin-orbit coupling for neutral atoms

Reviews: J. Dalibard, F. Gerbier, G. Juzeliunas, P. Ohberg, Rev. Mod. Phys. (2011)
N Goldman, G. Juzeliunas, P. Ohberg, I. B. Spielman, Rep. Prog. Phys. (2014)

Topological states using synthetic gauge potentials

- Synthetic **magnetic field** ⟶ The quantum Hall effect with cold atoms!

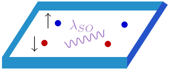


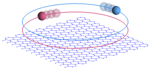


$$\sigma_H = 1/\rho_H = \nu \times (e^2/h)$$

- Synthetic **spin-orbit coupling**

⟶ The quantum spin Hall effect, 2D/3D topological insulators





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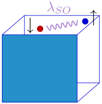

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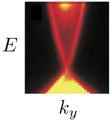


$$E \sim vk$$

Xia et al. Nat. Phys. 2009

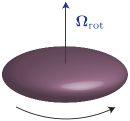
- Synthetic **spin-orbit coupling** (+ Zeeman splitting and s-wave interactions)

⟶ topological superconductivity with Majorana modes

General overview of the schemes considered (so far) in experiments...

Ref: N. Goldman et al., Rep. Prog. Phys. (2014)

- **Rotation**

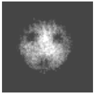


Hamiltonian in the rotating frame:

$$\hat{H} = \frac{1}{2m}\left(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})\right)^2 + V(\boldsymbol{x}) + W_{\text{anti-trap}}(\boldsymbol{x}) + \ldots$$

$$\boldsymbol{A} = m\,\boldsymbol{\Omega}_{\text{rot}} \times \boldsymbol{x} \longrightarrow \boxed{q\boldsymbol{B} = 2m\,\boldsymbol{\Omega}_{\text{rot}}}$$

Ref: N. Cooper, Adv. Phys. '08



Dalibard et al. '00

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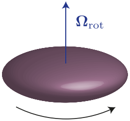

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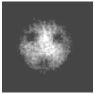
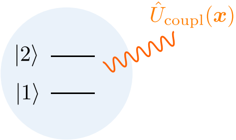


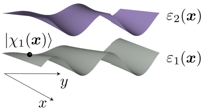
Dalibard et al. '00

- **Raman dressing**

The **adiabatic motion** of atoms in a ***dressed state*** (local energy eigenstates)



atom–light coupling



Berry connection

$$\longrightarrow \boxed{\boldsymbol{A}(\boldsymbol{x}) = i\hbar\langle\chi_1|\boldsymbol{\nabla}\chi_1\rangle}$$

$\longrightarrow$ **synthetic magnetic fields,**
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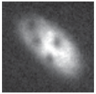

Spielman et al. '09

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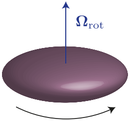

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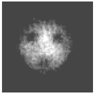

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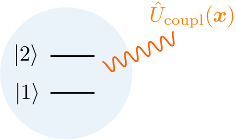

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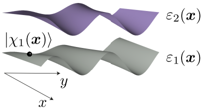

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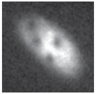
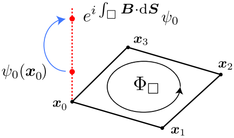


Spielman et al. '09

$\longrightarrow$ **synthetic magnetic fields,**
synthetic spin-orbit coupling, ...

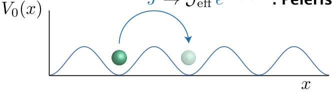
- **Mimic the Aharonov–Bohm phase in optical lattices**

Induce **complex tunneling** matrix elements



$\longrightarrow$

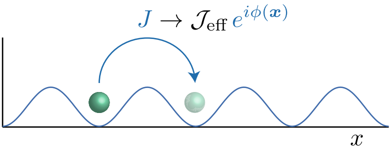
$J \to \mathcal{J}_{\rm eff}\, e^{i\phi(\boldsymbol{x})}$ **: Peierls phase-factors**



$\longrightarrow$ **synthetic magnetic flux in lattices,**
lattice gauge theory (e.g. non-Abelian),
synthetic spin-orbit coupling, ...

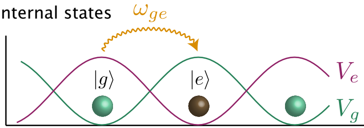
Different ways to induce/control the hopping in optical lattices

- The **goal**: $V_0(x)$



Reviews:
Dalibard, Gerbier, Juzeliunas, Ohberg, RMP '10
Goldman, Juzeliunas, Ohberg, Spielman, RPP '14

- **Option 1**: Use the internal states

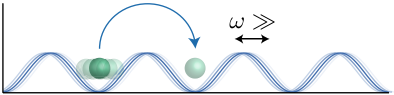


Theory (proposals):

Jaksch & Zoller, NJP '03

Gerbier & Dalibard, NJP '10

- **Option 2**: Shake!

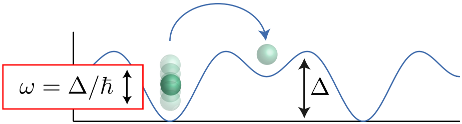


Experiments (since 2011):

Struck, Eckardt, Sengstock, Lewenstein et al. (Hamburg)

Jotzu, Esslinger et al. (Zurich)

- **Option 3**: Combine superlattices and resonant time-modulations



Experiments (since 2011):

Aidelsburger, Bloch et al. (MPQ)

Miyake, Ketterle et al. (MIT)

Outline

Part 1: Shaking atoms!

- Generating effective Hamiltonians: “Floquet” engineering
- Topological matter by shaking atoms
- Some final remarks about energy scales

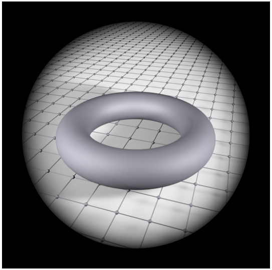
Part 2: Seeing topology in the lab!

- Loading atoms into topological bands
- Anomalous velocity and Chern-number measurements
- Seeing topological edge states with atoms

Part 3: Using internal atomic states!

- Cold Atoms = moving 2-level systems
- Internal states in optical lattices: laser-induced tunneling
- Synthetic dimensions: From 2D to 4D quantum Hall effects

Part 1: Shaking atoms!



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The general picture : A static system is modulated periodically in time

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t), \quad \hat{V}(t+T) = \hat{V}(t), \quad T = 2\pi/\omega \text{ : the period}$$

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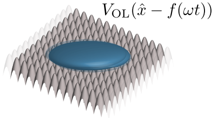
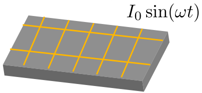
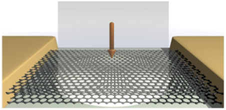
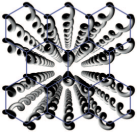






From: Suarez Morrel and Foa Torres, PRB 2012



From: Rechtsman et al., Nature 2013

$\hat{H}_0$	$\hat{V}(t)$
Cold atoms in optical lattices	Shaking the lattice, Modulating the hopping (lattice depth), Time-dependent magnetic fields, Additional lasers, ...
Cold atoms on the surface of a chip	Modulating the currents,...
Electrons in a material (ex: graphene, semiconductors,...)	Radiation, mechanical deformation,...
Light in photonic crystals	Helical waveguides (time=a spatial direction)

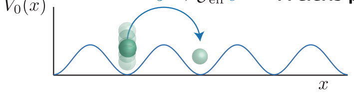
Refs: Cayssol, Dora, Simon and Moessner (Phys. Status Solidi RRL 2013),
M. Polini, F. Guinea, M. Lewenstein, H. C. Manoharan and V. Pellegrini (Nat. Nanotech. 2013).

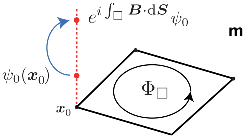
Ref: I. Carusotto and C. Ciuti (Rev. Mod. Phys. 2013).

Our goal: designing topological models by shaking atoms

- The **basic concept**:

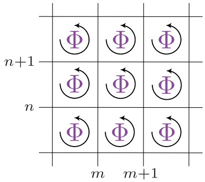
$J \rightarrow \mathcal{J}_{\text{eff}}\, e^{i\phi(\boldsymbol{x})}$ **: Peierls phase-factors**





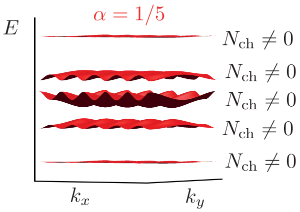
mimic the A-B effect

- Example 1: The **Harper-Hofstadter** model



$$\hat{H} = -J \sum_{m,n} \hat{a}^{\dagger}_{m,n+1}\hat{a}_{m,n} + e^{i2\pi\alpha n}\hat{a}^{\dagger}_{m+1,n}\hat{a}_{m,n} + \text{H.c.}$$

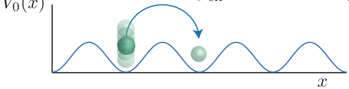
$\alpha = \Phi/\Phi_0$: uniform flux per plaquette (in units of flux quantum)

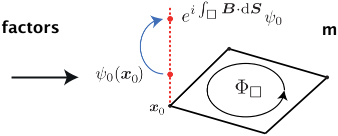


Our goal: designing topological models by shaking atoms

- The **basic concept**:

$J \rightarrow \mathcal{J}_{\text{eff}}\, e^{i\phi(\boldsymbol{x})}$ **: Peierls phase-factors**





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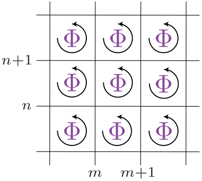

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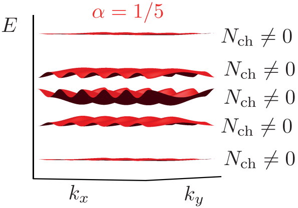
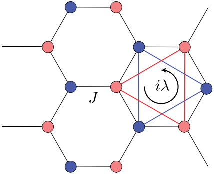
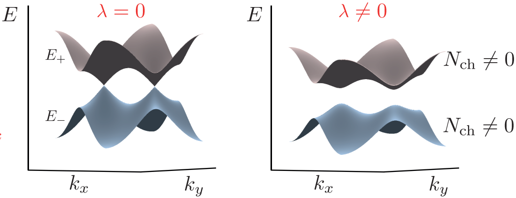


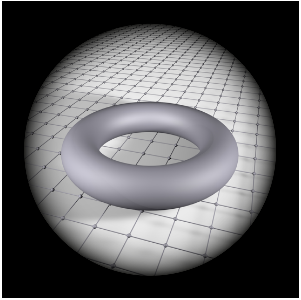
- Example 2: The **Haldane** model



$$\hat{H} = -J \sum_{\langle j,k \rangle} \hat{a}^{\dagger}_{j}\hat{a}_{k} + \lambda \sum_{\langle\langle j,k \rangle\rangle} i^{\circlearrowleft}\hat{a}^{\dagger}_{j}\hat{a}_{k}$$



Generating effective Hamiltonians: “Floquet” engineering



The central notion : the effective *time-independent* Hamiltonian

- A static system $\hat{H}_0$ is modulated periodically in time

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t), \quad \hat{V}(t+T) = \hat{V}(t), \quad T = 2\pi/\omega \text{ : the period}$$

- Generally, one adopts a stroboscopic view [$T \ll t_{\text{charact}}$] : $t = NT$, $N \in \mathbb{N}$

$$|\psi(t = NT)\rangle = \left[\hat{U}(T)\right]^N |\psi_0\rangle = \left[\mathcal{T} e^{-i\int_0^T \hat{H}(\tau)\mathrm{d}\tau}\right]^N |\psi_0\rangle$$

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- Driving is interesting : $\hat{H}_0$ (“normal”) $\to$ $\hat{\mathcal{H}}_{\text{eff}}$ (potentially) **Super !**

- Tuning $\hat{V}(t)$: A versatile tool to engineer **gauge fields**, **topological bands**, ...

$$\hat{\mathcal{H}}_{\text{eff}} = \frac{1}{2m}\left[\left(\hat{p}_x + \hat{A}_x\right)^2 + \left(\hat{p}_y + \hat{A}_y\right)^2\right] + \ldots$$

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- In general, the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ cannot be derived exactly...

$$e^{-iT\hat{\mathcal{H}}_{\text{eff}}} = \mathcal{T} e^{-i\int_0^T \hat{H}(\tau)\mathrm{d}\tau} = \dots ?$$

The effective Hamiltonian and the Magnus expansion

- We want to evaluate the time-evolution operator between times t_0 and t_f

$$\hat{U}(t_f; t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t_f} \hat{H}(\tau)\mathrm{d}\tau\right), \qquad \hat{H}(t+T) = \hat{H}(t).$$

- Stroboscopic evolution (i.e. neglect micro-motion) : $t_f = t_0 + NT$ with $N \in \mathbb{N}$

$$\hat{U}(t_f; t_0) = [\hat{U}(t_0 + T; t_0)]^N, \quad \text{where} \quad \hat{U}(t_0 + T; t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t_0+T} \hat{H}(\tau)\mathrm{d}\tau\right)$$

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- The time-ordered integral can be expanded through the Magnus formula

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- Setting $\hat{U}(t_0+T;t_0) = e^{-iT\hat{H}_{\mathsf{F}}}$, the effective Hamiltonian is given by the series

$$\hat{H}_{\mathsf{F}} = (1/T)\int_{t_0}^{T+t_0}\hat{H}(t)\mathrm{d}t - \frac{i}{2T}\int_{t_0}^{t_0+T}\int_{t_0}^{t}[\hat{H}(t),\hat{H}(\tau)]\mathrm{d}\tau\mathrm{d}t + \dots \qquad (1)$$

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- If we expand $\hat{H}(t)$ into its Fourier components,

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) = \hat{H}_0 + \sum_{j\neq 0}\hat{V}^{(j)}\exp(ij\omega t),$$

  and perform the integrals in Eq. 1, we obtain the equivalent expression

$$\hat{H}_\mathsf{F} = \hat{H}_0 + \frac{1}{\omega}\sum_{j>0}\frac{1}{j}\left\{\left[\hat{V}^{(+j)},\hat{V}^{(-j)}\right] - e^{ij\omega t_0}\left[\hat{V}^{(+j)},\hat{H}_0\right] + e^{-ij\omega t_0}\left[\hat{V}^{(-j)},\hat{H}_0\right]\right\} + \dots$$

The effective Hamiltonian and the Magnus expansion

- The effective Hamiltonian is given by a perturbative expansion in powers of $(1/\omega)$:

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- Useful to calculate effective Hamiltonians in the high-frequency regime $\omega \gg$!

- Useful to identify interesting time-modulated (cold-atom) setups [i.e. $\hat{H}_0$, $\hat{V}(t)$] !

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- Useful to calculate effective Hamiltonians in the high-frequency regime $\omega \gg$!
- Useful to identify interesting time-modulated (cold-atom) setups [i.e. $\hat{H}_0,\, \hat{V}(t)$] !
- Several issues and subtleties should be addressed :
  - The effective Hamiltonian $\hat{H}_{\mathsf{F}}(t_0)$ explicitly depends on the initial time t_0...

    $\longrightarrow$ What is the role of t_0-terms ?
  - Is micro-motion really irrelevant ? How can this be evaluated ?
  - Is the convergence of the series guaranteed ? What if $\hat{H}_0, \hat{V}^{(j)} \sim \omega$?

The t_0-dependent terms : a simple illustration

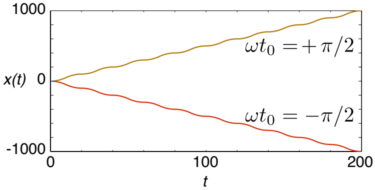
- Consider a particle driven by a time-modulated force F :

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) = \frac{\hat{p}^2}{2m} + F\cos(\omega t)\hat{x}$$

- The Magnus expansion provides the (exact) effective Hamiltonian

$$\hat{H}_{\mathrm{F}}(t_0) = \frac{1}{2m}\left[\hat{p} + \mathcal{A}(t_0)\right]^2 + \text{cst}, \quad \text{where } \mathcal{A}(t_0) = \frac{F}{\omega}\sin(\omega t_0).$$

- The driving only modifies the **initial** mean velocity : $v(t_0) \to v(t_0) + \mathcal{A}(t_0)/m$



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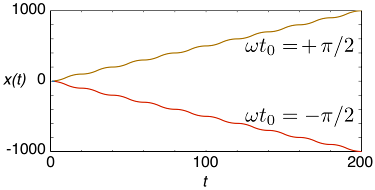
- The t_0-dependent terms can be removed by a **unitary (gauge) transformation**

$$\hat{H}_{\mathsf{F}}(t_0) = \hat{S}^\dagger(t_0)\hat{H}_0\hat{S}(t_0) \quad \text{where } \hat{S}(t_0) = \exp\left[i\mathcal{A}(t_0)\hat{x}\right]$$

$$\longrightarrow \hat{U}(T + t_0; t_0) = e^{-iT\hat{H}_{\mathsf{F}}(t_0)} = \hat{S}^\dagger e^{-iT\hat{H}_0}\hat{S}$$



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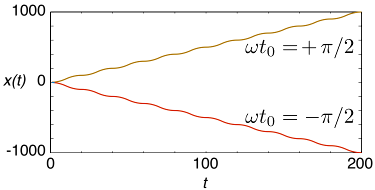


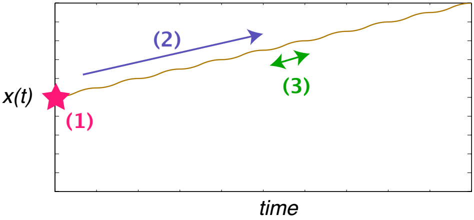
After a long time: $t_f = t_0 + NT$

$$|\psi(t_f)\rangle = \hat{S}^\dagger e^{-iNT\hat{H}_0}\hat{S}|\psi_0\rangle$$

$\hat{S}(t_0) = \exp\left[i\mathcal{A}(t_0)\hat{x}\right]$: initial kick

- In general, there are three **distinct** notions :
  - (1) The **initial kick** related to the **initial phase** of the modulation
  - (2) The long-time dynamics ruled by an **effective Hamiltonian** $\hat{H}_{\text{eff}} \neq \hat{H}_{\text{F}}(t_0)$
  - (3) The **micro-motion** (i.e. what happens within a period)

$$\psi(t_f) = \underbrace{\hat{M}(t_f)}_{(3)} \underbrace{e^{-i(t_f - t_i)\hat{H}_{\text{eff}}}}_{(2)} \underbrace{\hat{S}(t_0)}_{(1)} \psi(t_0)$$



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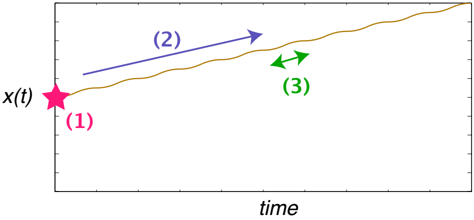

- We can formally **separate** these effects by using a **unitary transformation**

$$\psi(t) \to \phi(t) = e^{i\hat{K}(t)}\psi(t), \quad i\partial_t \phi(t) = \hat{H}_{\text{eff}}\phi(t),$$

$$\psi(t_f) = \hat{U}(t_0 \to t_f)\psi(t_i) = e^{-i\hat{K}(t_f)} e^{-i(t_f - t_0)\hat{H}_{\text{eff}}} e^{i\hat{K}(t_0)} \psi(t_0)$$

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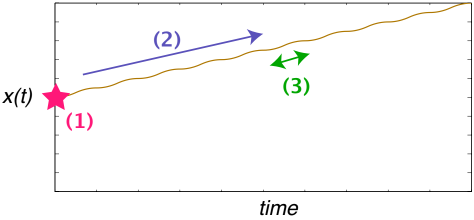

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- Question : Is it possible to compute $\hat{H}_{\text{eff}}$ and $\hat{K}(t)$ explicitly ?

Deriving the effective Hamiltonian [see Rahav et al. PRA '03, Goldman-Dalibard PRX '14]

- We consider the time-dependent unitary transformation

$$i\partial_t \psi(t) = \hat{H}(t)\psi(t), \qquad \psi(t) \to \phi(t) = e^{i\hat{K}(t)}\psi(t),$$

- In the new frame the Hamiltonian is imposed to be **time-independent** :

$$i\partial_t \phi(t) = \hat{H}_{\text{eff}}\phi(t)$$

- The relation between $\hat{H}(t)$ and $\hat{H}_{\text{eff}}$ is given by the usual transformation

$$\hat{H}_{\text{eff}} = e^{i\hat{K}(t)}\hat{H}(t)e^{-i\hat{K}(t)} + i\left(\frac{\partial e^{i\hat{K}(t)}}{\partial t}\right)e^{-i\hat{K}(t)} \qquad (*)$$

- We expand $\hat{H}_{\text{eff}}$ and $\hat{K}(t)$ in powers of $1/\omega$

$$\hat{H}_{\text{eff}} = \sum_{n=0}^{\infty}\frac{1}{\omega^n}\hat{H}_{\text{eff}}^{(n)}, \quad \hat{K}(t) = \sum_{n=1}^{\infty}\frac{1}{\omega^n}\hat{K}^{(n)}$$

$\longrightarrow$ insert into Eq. $(*)$ to get $\hat{H}_{\text{eff}}^{(0)}, \hat{H}_{\text{eff}}^{(1)}, \hat{H}_{\text{eff}}^{(2)}, \ldots$ and $\hat{K}^{(1)}, \hat{K}^{(2)}, \ldots$

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$$\hat{H}_{\text{eff}} = e^{i\hat{K}(t)}\hat{H}(t)e^{-i\hat{K}(t)} + i\left(\frac{\partial e^{i\hat{K}(t)}}{\partial t}\right)e^{-i\hat{K}(t)} \qquad (*)$$

- We expand $\hat{H}_{\text{eff}}$ and $\hat{K}(t)$ in powers of $1/\omega$

$$\hat{H}_{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{\omega^n}\hat{H}_{\text{eff}}^{(n)}, \quad \hat{K}(t) = \sum_{n=1}^{\infty} \frac{1}{\omega^n}\hat{K}^{(n)}$$

$\longrightarrow$ insert into Eq. $(*)$ to get $\hat{H}_{\text{eff}}^{(0)}, \hat{H}_{\text{eff}}^{(1)}, \hat{H}_{\text{eff}}^{(2)}, \ldots$ and $\hat{K}^{(1)}, \hat{K}^{(2)}, \ldots$

- We then have the full time-evolution :

$$\psi(t_f) = \hat{U}(t_0 \to t_f)\psi(t_i) = e^{-i\hat{K}(t_f)}e^{-i(t_f - t_0)\hat{H}_{\text{eff}}}e^{i\hat{K}(t_0)}\psi(t_0)$$

General formulas [see Goldman-Dalibard PRX '14]

- For a general time-periodic problem

$$\hat{H}(t) = \hat{H}_0 + \sum_{j=1}^{\infty} V^{(j)} e^{ij\omega t} + V^{(-j)} e^{-ij\omega t},$$

the long-time dynamics is well-captured by the **effective Hamiltonian** :

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{\omega}\sum_{j=1}^{\infty}\frac{1}{j}[V^{(j)}, V^{(-j)}] + \frac{1}{2\omega^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left([[V^{(j)}, \hat{H}_0], V^{(-j)}] + \text{h.c.}\right)$$
$$+ \frac{1}{3\omega^2}\sum_{j,l=1}^{\infty}\frac{1}{jl}\left([V^{(j)}, [V^{(l)}, V^{(-j-l)}]] - [V^{(j)}, [V^{(-l)}, V^{(l-j)}]] + \text{h.c.}\right) + \dots,$$

$\longrightarrow$ good basis to **identify** schemes leading to **topological properties** !

- The micro-motion + initial-kick effects are well described by the **kick operator** :

$$\hat{K}(t) = \frac{1}{i\omega}\sum_{j=1}^{\infty}\frac{1}{j}\left(V^{(j)} e^{ij\omega t} - V^{(-j)} e^{-ij\omega t}\right) + \dots$$

$\longrightarrow$ good basis to **estimate** the effects due to **micro-motion** on observables !

Dealing with the convergence of the series [Goldman, Dalibard et al., PRA '15]

- The perturbative approach works fine if $\hat{H}(t)$ remains finite for $\omega \to \infty$
- However, we might deal with systems of the form

$$\hat{H}(t) = \hat{H}_{\text{regular}}(t) + \omega \hat{O}(t),$$

Examples : strong-driving regime, static (resonant) energy offset $\Delta = \hbar\omega$

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- Solution : perform a **unitary transformation** that removes all diverging terms !

$$|\psi\rangle \to |\psi'\rangle = \hat{R}(t)|\psi\rangle,\ \hat{R}(t) = \mathcal{T}\exp\left\{i\omega \int_0^t \hat{O}(\tau)\mathrm{d}\tau\right\}$$

$$\hat{H}(t) \to \hat{\mathcal{H}}(t) = \hat{R}(t)\hat{H}(t)\hat{R}^\dagger(t) - i\hat{R}(t)\partial_t\hat{R}^\dagger(t) = \hat{R}(t)\hat{H}_{\text{regular}}\hat{R}^\dagger(t)$$

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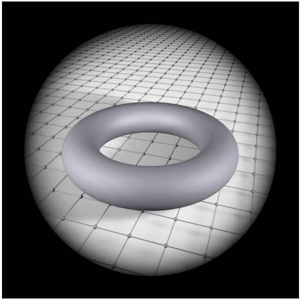
- If $\hat{R}(t)$ and $\hat{\mathcal{H}}(t)$ can be computed explicitly, i.e. $[\hat{O}(t), \hat{O}(t')] = 0$, then we are fine :

$$\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_0 + \sum_{j=1}^{\infty} \hat{\mathcal{V}}^{(j)} e^{ij\omega t} + \hat{\mathcal{V}}^{(-j)} e^{-ij\omega t} \text{ : is regular in the limit } \omega \to \infty$$

and we can apply our formula for the effective Hamiltonian (in the moving frame) :

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{\mathcal{H}}_0 + \frac{1}{\omega}\sum_{j=1}^{\infty}\frac{1}{j}[\hat{\mathcal{V}}^{(j)}, \hat{\mathcal{V}}^{(-j)}] + \frac{1}{2\omega^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left([[\hat{\mathcal{V}}^{(j)}, \hat{\mathcal{H}}_0], \hat{\mathcal{V}}^{(-j)}] + \text{h.c.}\right)\ldots,$$

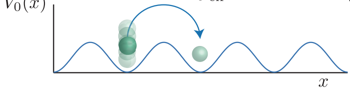
Topological "Floquet" matter by shaking atoms

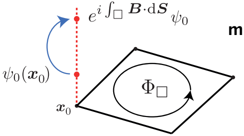


Our goal: designing topological models by shaking atoms

- The **basic concept**:

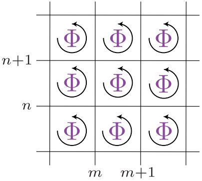
$J \rightarrow \mathcal{J}_{\text{eff}}\, e^{i\phi(\boldsymbol{x})}$ **: Peierls phase-factors**





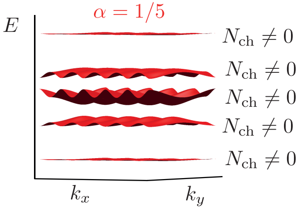
mimic the A–B effect

- Example 1: The **Harper–Hofstadter** model

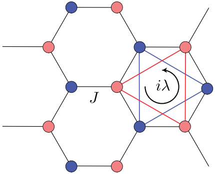


$$\hat{H} = -J \sum_{m,n} \hat{a}^{\dagger}_{m,n+1}\hat{a}_{m,n} + e^{i2\pi\alpha n}\hat{a}^{\dagger}_{m+1,n}\hat{a}_{m,n} + \text{H.c.}$$

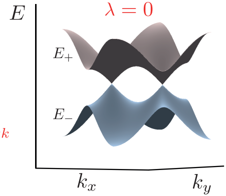
$\alpha = \Phi/\Phi_0$: uniform flux per plaquette (in units of flux quantum)

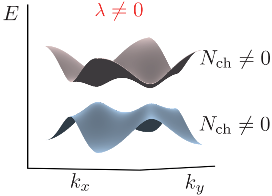


- Example 2: The **Haldane** model



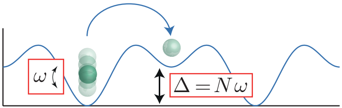
$$\hat{H} = -J \sum_{\langle j,k \rangle} \hat{a}^{\dagger}_{j}\hat{a}_{k} + \lambda \sum_{\langle\langle j,k \rangle\rangle} i^{\circlearrowleft}\hat{a}^{\dagger}_{j}\hat{a}_{k}$$





Useful example ... in view of creating fluxes in optical lattices

- Let us simplify the problem of the time-modulated superlattice (N integer)



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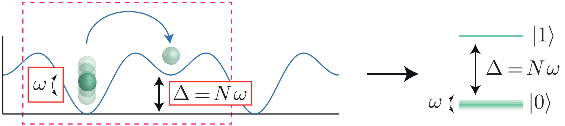

- We write the Hamiltonian of the two-level system as

$$\hat{H}(t) = J\,(|0\rangle\langle 1| + |1\rangle\langle 0|) + N\omega|1\rangle\langle 1| + \kappa\cos(\omega t + \phi)|0\rangle\langle 0|, \quad J \ll \omega$$

- In the strong-driving regime, $\kappa = K_0\omega$ with $K_0 \sim 1$: two diverging terms !

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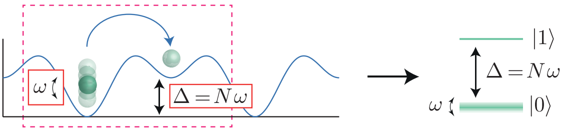

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$$\rightarrow \hat{\mathcal{H}}(t) = J|0\rangle\langle 1| \sum_{j=-\infty}^{\infty} e^{ij\omega t}\mathcal{J}_{N+j}(K_0)e^{i(j+N)\phi} + \text{h.c.} \qquad \left[e^{ix\sin(y)} = \sum_{n=-\infty}^{\infty}\mathcal{J}_n(x)e^{iny}\right]$$

- To lowest order, the effective Hamiltonian is given by

$$\hat{\mathcal{H}}_{\text{eff}} \approx \hat{\mathcal{H}}_0 = J\mathcal{J}_N(K_0)e^{iN\phi}\,|0\rangle\langle 1| + \text{h.c.}$$

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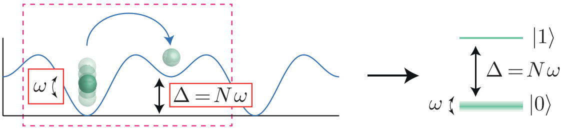

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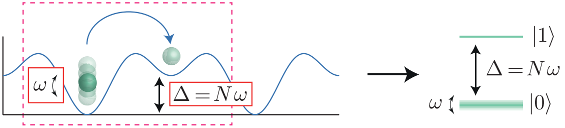

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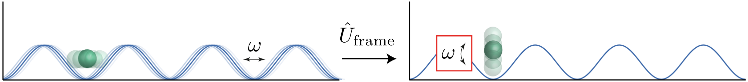
- $N = 1 \rightarrow \hat{\mathcal{H}}_{\text{eff}} \approx J\mathcal{J}_1(K_0)e^{i\phi}\,|0\rangle\langle 1| + \text{h.c.}$: the effective coupling is **complex** !

The shaken optical lattices

- Let us now consider the full 1D shaken lattice without offset ($\Delta = 0$)



$$\tilde{H}(t) = \frac{\hat{p}^2}{2m} + U_{\rm opt}[\hat{x} - x_0(t)] \qquad \hat{H}(t) = \frac{\hat{p}^2}{2m} + U_{\rm opt}(\hat{x}) - F(t)\hat{x}$$

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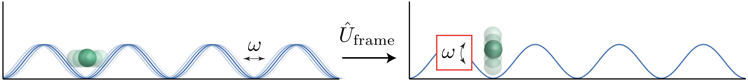

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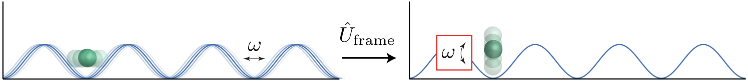

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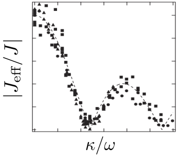

Lignier, Arimondo et al. 2007

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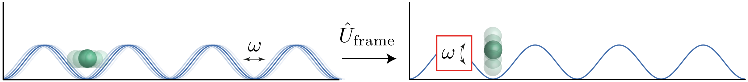

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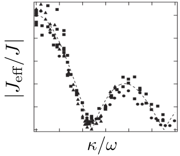
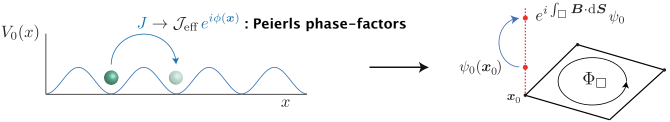


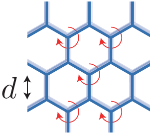
Lignier, Arimondo et al. 2007

- Our goal is to create some fluxes in 2D... impossible by shaking the lattice ?



Shaking a 2D optical lattices circularly

- We consider a 2D honeycomb lattice, shaken circularly, in the moving frame



$$\hat{H}(t) = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + U_{\text{honey}}(\boldsymbol{x}) - \boldsymbol{F}(t) \cdot \hat{\boldsymbol{x}}$$

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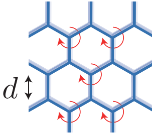

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- The tight-binding Hamiltonian : NN tunneling + shaking

$$\hat{H}(t) = -J \sum_{\langle j,k \rangle} \hat{a}_j^\dagger \hat{a}_k - \sum_j \boldsymbol{F}(t) \cdot \boldsymbol{r}_j \, \hat{a}_j^\dagger \hat{a}_j,$$

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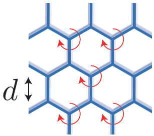

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- For a strong-driving amplitude $\kappa = Fd \sim \omega$, we perform a unitary transformation :

$$\hat{R}(t) = \exp\left\{ i(F/\omega) \sum_j \boldsymbol{r}_j \cdot \left[\sin(\omega t)\boldsymbol{e}_x - \cos(\omega t)\boldsymbol{e}_y\right] \hat{a}_j^\dagger \hat{a}_j \right\}$$

$$\rightarrow \hat{\mathcal{H}}(t) = \sum_{n=-\infty}^{\infty} \hat{\mathcal{H}}^{(n)} e^{in\omega t}, \quad \hat{\mathcal{H}}^{(n)} = -J \mathcal{J}_n(\kappa/\omega) \sum_{\langle j,k \rangle} \hat{a}_j^\dagger \hat{a}_k e^{-in\theta_{jk}}$$

where we have introduced the link-angles : $\boldsymbol{r}_j - \boldsymbol{r}_k = d\left[\cos(\theta_{jk})\boldsymbol{e}_x + \sin(\theta_{jk})\boldsymbol{e}_y\right]$

Shaking a 2D optical lattices circularly

- We have the time-dependent Hamiltonian in the moving frame

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- To lowest order : nothing very special...

$$\hat{\mathcal{H}}_{\text{eff}} \approx \hat{\mathcal{H}}^{(0)} = -J\mathcal{J}_0(\kappa/\omega) \sum_{\langle j,k \rangle} \hat{a}_j^\dagger \hat{a}_k$$

: the NN tunneling is renormalized

Shaking a 2D optical lattices circularly

- We have the time-dependent Hamiltonian in the moving frame

$$\hat{\mathcal{H}}(t) = \sum_{n=-\infty}^{\infty} \hat{\mathcal{H}}^{(n)} e^{in\omega t}, \quad \hat{\mathcal{H}}^{(n)} = -J\mathcal{J}_n(\kappa/\omega) \sum_{\langle j,k \rangle} \hat{a}_j^\dagger \hat{a}_k e^{-in\theta_{jk}}$$

- We can calculate the effective Hamiltonian

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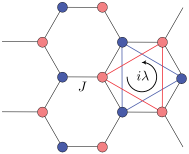
: the NN tunneling is renormalized

- The first correction to $\hat{\mathcal{H}}_{\text{eff}}$: NNN complex tunneling terms !

$$\frac{1}{\omega} \sum_{j=1}^{\infty} \frac{1}{j} [\hat{\mathcal{H}}^{(j)}, \hat{\mathcal{H}}^{(-j)}] \approx \frac{\sqrt{3}J^2}{\omega} \mathcal{J}_1^2(\kappa/\omega) \sum_{\langle\langle j,k \rangle\rangle} \hat{a}_j^\dagger \hat{a}_k \, e^{\pm i\pi/2}$$

- The effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}$ corresponds to the Haldane model !



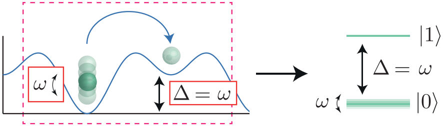
$$\hat{H} = -J \sum_{\langle j,k \rangle} \hat{a}_j^\dagger \hat{a}_k + \lambda \sum_{\langle\langle j,k \rangle\rangle} i^{\circlearrowleft} \hat{a}_j^\dagger \hat{a}_k$$

This experiment was realized at ETH Zurich in the group of T. Esslinger

Ref: Jotzu et al. Nature 2014

Combining superlattices and resonant modulation

- We have seen that **resonant driving** naturally leads to complex coupling elements



$$\hat{\mathcal{H}}_{\mathrm{eff}} \approx J \mathcal{J}_1(K_0) e^{i\phi} |0\rangle\langle 1| + \mathrm{h.c.}$$

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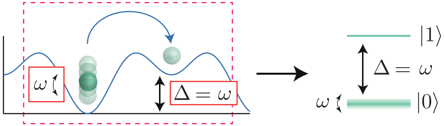

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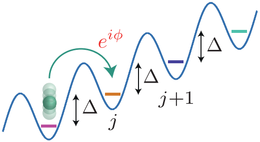
- Kolovsky's idea [EPL '11] : Wannier-Stark-ladder + resonant modulation



1D: $\hat{\mathcal{H}}_{\text{eff}} \approx J\mathcal{J}_1(K_0)\sum_j \hat{a}^\dagger_{j+1}\hat{a}_j e^{i\phi} + \text{h.c.}$

2D: $\hat{\mathcal{H}}_{\text{eff}} \approx J\mathcal{J}_1(K_0)\sum_{j,k} \hat{a}^\dagger_{j+1,k}\hat{a}_{j,k} e^{i\phi(k)} + J\sum_{j,k} a^\dagger_{j,k+1}\hat{a}_{j,k} + \text{h.c.}$

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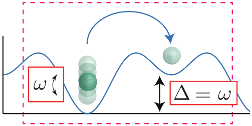

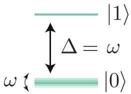

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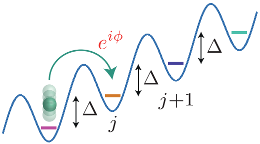

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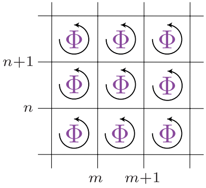
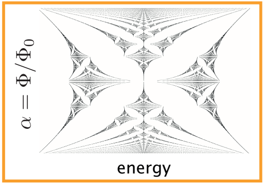
- This system would be equivalent to the **Harper-Hofstadter** model



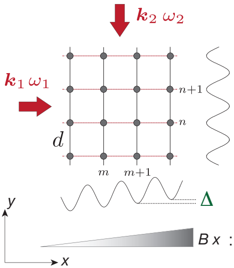
$$\hat{H} = -J\sum_{m,n} \hat{a}^\dagger_{m,n+1}\hat{a}_{m,n} + e^{i2\pi\alpha n}\hat{a}^\dagger_{m+1,n}\hat{a}_{m,n} + \text{H.c.}$$

$\alpha = \Phi/\Phi_0$: uniform flux per plaquette (in units of flux quantum)



The Munich experiment by Aidelsburger et al. PRL '13 [see also Miyake et al. PRL '13]

- The **Wannier-Stark ladder** is created by a magnetic field gradient



Lattice modulation: 2 running waves

$$E_1(\boldsymbol{r},t) \sim e^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}+\omega_1 t)}$$

$$E_2(\boldsymbol{r},t) \sim e^{i(\boldsymbol{k}_2\cdot\boldsymbol{r}+\omega_2 t)}$$

$$\longrightarrow \quad V_{\text{mod}}(\boldsymbol{r},t) = 2\kappa\cos(\omega t + \boldsymbol{q}\cdot\boldsymbol{r})$$

$$\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_2$$

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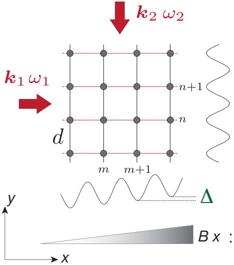

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- Setting $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = \pi/2d$, the Hamiltonian reads

$$\hat{H}(t) = -J\sum_{m,n}\hat{a}^\dagger_{m\pm1,n}\hat{a}_{m,n}+\hat{a}^\dagger_{m,n\pm1}\hat{a}_{m,n}+\sum_{m,n}\hat{n}_{m,n}\left\{\omega m + 2\kappa\cos\left[\omega t + \frac{\pi}{2}(m+n)\right]\right\}$$

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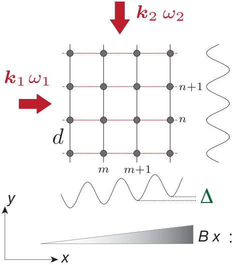

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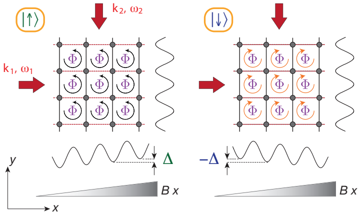
- The effective Hamiltonian is given by the Harper-Hofstadter form

$$\hat{\mathcal{H}}_{\text{eff}} = -\sum_{m,n} J_x\hat{a}^\dagger_{m+1,n}\hat{a}_{m,n}e^{i\Phi n} + J_y\hat{a}^\dagger_{m,n+1}\hat{a}_{m,n} + \text{h.c.},$$

$$J_x = J\mathcal{J}_1\left(\frac{2\sqrt{2}\kappa}{\omega}\right), J_y = J\mathcal{J}_0\left(\frac{2\sqrt{2}\kappa}{\omega}\right), \quad \Phi = \frac{\pi}{2}(= q_y d)$$

Schemes leading to spin-orbit coupling ? Physics of topological insulators ?

- The Munich setup : two internal states $|\uparrow, \downarrow\rangle$ with **opposite** magnetic moment



The **synthetic gauge potential** equivalent to a **spin–orbit–coupling** effect:

$$\boldsymbol{A}_{\text{eff}} = (-B_{\text{eff}}\, y, 0, 0) \longrightarrow \boxed{\boldsymbol{A}_{\text{eff}} = \hat{\sigma}_z(-B_{\text{eff}}\, y, 0, 0)}$$

It is the form required to observe the **quantum spin Hall effect**

[see Bernevig & Zhang PRL '06; also Kane & Mele PRL '05]

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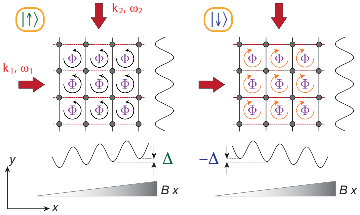

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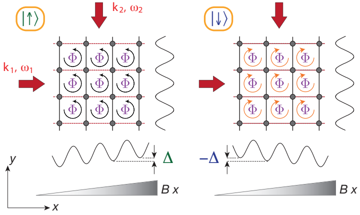

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$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{\omega}\sum_{j=1}^{\infty}\frac{1}{j}[\hat{V}^{(j)}, \hat{V}^{(-j)}] + \dots$$

$$= \hat{H}_0 + \frac{i}{2\omega}[\hat{A}, \hat{B}] + \dots$$

- A possible solution : $\hat{H}_0 = \frac{\hat{p}^2}{2m}, \hat{A} = \frac{\hat{p}_x^2 - \hat{p}_y^2}{2m}, \hat{B} = \kappa(\hat{x}\hat{\sigma}_x - \hat{y}\hat{\sigma}_y)$

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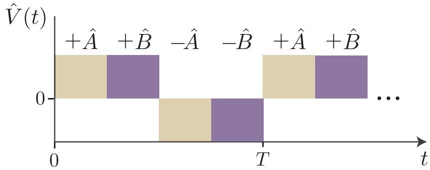
- In practice ? We approximate the driving $\hat{A}\cos(\omega t) + \hat{B}\sin(\omega t)$ by square waves

$$\hat{H}(t) = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} +$$



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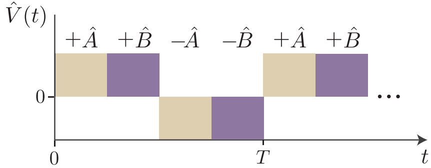

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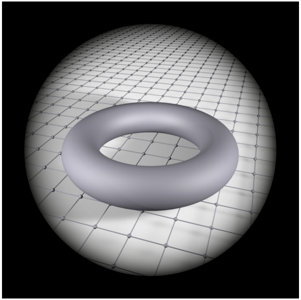
- **Physical realization** : a pulsed optical lattice with space-dependent magnetic field $\hat{H}_0 = \hat{T}_x + \hat{T}_y$ (where $T_{x,y}$: hopping terms $\sim J$, with $J \ll \omega$ bounded !)

$$\text{sequence : } \left\{ 2\hat{T}_x\,,\; \hat{H}_0 + \kappa(\hat{x}\hat{\sigma}_x - \hat{y}\hat{\sigma}_y)\,,\; 2\hat{T}_y\,,\; \hat{H}_0 - \kappa(\hat{x}\hat{\sigma}_x - \hat{y}\hat{\sigma}_y) \right\}$$

Some final remarks about energy scales



Some remarks about energy scales, temperature requirements

- Consider a standard optical lattice (retro-reflected laser light)

$$V(x) = U_0 \cos^2(kx), \quad \text{laser wavelength} : \lambda = 2\pi/k, \; \text{lattice spacing} : d = \lambda/2$$

- The energy scales on the lattice are set by the **recoil energy**

$$E_\mathrm{R} = \frac{\hbar^2 k^2}{2m} = \frac{h^2}{8md^2}, \qquad E_\mathrm{R}/h \sim 10\,\mathrm{kHz} \longrightarrow E_\mathrm{R}/k_\mathrm{B} \sim 100\,\mathrm{nK}$$

- The amplitude of tunneling matrix elements is well approximated by

$$J = \frac{4E_\mathrm{R}}{\sqrt{\pi}} \left(\frac{U_0}{E_\mathrm{R}}\right)^{3/4} \exp\left[-2\left(\frac{U_0}{E_\mathrm{R}}\right)^{1/2}\right]$$

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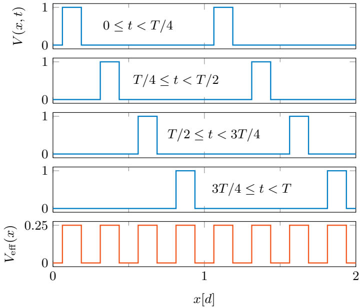

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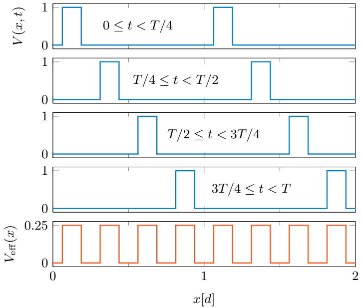


- In practice : we propose to use a moving spin-dependent lattice

$$V(x,t) = V_{\text{L}} \cos(2kx - \omega t)\hat{\sigma}_z + V_{\text{B}} \cos(N\omega t)\hat{\sigma}_x, \quad N \in \mathbb{N}$$

$$\hat{H}_{\text{eff}} = \frac{p^2}{2m} + \frac{U_{\text{eff}}}{2}\cos(2Nkx)\hat{\sigma}_x, \quad U_{\text{eff}} = \mathcal{J}_N\left(\frac{2V_{\text{L}}}{\hbar\omega}\right)V_{\text{B}},$$

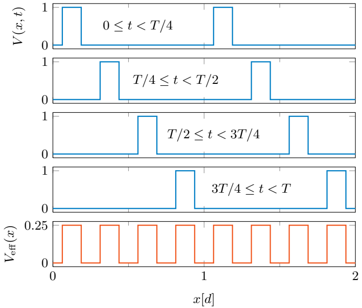
- This generates a lattice of spacing $d_{\text{new}} = d/N$, where $N \in \mathbb{N}$.

Some remarks about energy scales, temperature requirements

- Would it be possible to increase all energy scales ?

  Sub-wavelength lattices : $d_{\text{new}} \ll d = \lambda/2 \longrightarrow \tilde{E}_{\text{R}} = \frac{h^2}{8m(d_{\text{new}})^2} \gg E_{\text{R}} = \frac{h^2}{8md^2}$

- Idea based on time-modulated systems [Nascimbene et al., to appear in PRL '15]



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- This generates a lattice of spacing $d_{\text{new}} = d/N$, where $N \in \mathbb{N}$.
- Can be extended in 2D to create Chern bands...