T-duality II: Applications

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Abstract

This second part of two talks aims to apply T-Duality to open strings. A non-trivial example with intersecting branes leads to non-commutative geometry on the worldvolume of D-branes.

Contents

1 T-duality and open strings

As mentioned in the first talk, a boundary term has to be added if the worldsheet has $\partial\Sigma \neq \emptyset$ in order to restore gauge invariance; the sigma model reads

$$S = \frac{1}{4\pi\alpha'} \left\{ \int_\Sigma \left[G(X)_{ab}\, dX^a \wedge \star dX^b + B(X)_{ab}\, dX^a \wedge dX^b \right] + \int_{\partial\Sigma} A_a\, dX^a \right\} . \qquad (1.1)$$

The conventions for the worldsheet are as follows: We assume conformal gauge, i.e. the worldsheet metric takes the form $h = \mathrm{diag}(-1, 1)$. The coordinates on Σ are $\{\tau, \sigma\}$ with $\sigma \in [0, \pi]$ and the volume element is $d\tau \wedge d\sigma$. By the definition of the Hodge star $\alpha \wedge \star\beta = h(\alpha, \beta) d\tau \wedge d\sigma$ for α, β arbitrary n-forms we have in particular $\star d\tau = -d\sigma$ and $\star d\sigma = -d\tau$. The equations of motion remain the same, i.e.

$$d \star dX^a + \Gamma^a{}_{bc}\, dX^b \wedge \star dX^c = \tfrac{1}{2}\, G^{am}\, H_{mbc}\, dX^b \wedge dX^c\,, \qquad (1.2)$$

but are supplemented by the boundary conditions

$$\int_{\partial\Sigma} \left(G_{ab} \star dX^b + F_{ab}\, dX^b \right) \delta X^a = 0 \qquad (1.3)$$

with $F = B + dA$. If we assume that the variation and all the fields vanish at infinite times τ and use the conventions on the worldsheet one possible boundary condition is

$$\boxed{\left(\partial_\sigma X^a - G^{am} F_{mb}\, \partial_\tau X^b \right)\Big|_{\sigma=0,\pi} = 0} \qquad (1.4)$$

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with G^{ab} the components of the inverse metric. Of course, we can also have Dirichlet boundary conditions

$$\delta X^a|_{\sigma=0,\pi} = 0 \qquad \Longrightarrow \qquad \partial_\tau X^a|_{\sigma=0,\pi} = 0 \tag{1.5}$$

or mixtures of the two. We see that F interpolates between Neumann and Dirichlet boundary conditions.

The spacetime symmetries are

- B-field gauge transformations

$$B \to B + d\xi \qquad \text{with} \qquad A \to A - \xi \qquad \text{for} \quad \xi \in \Gamma(T^*M)\,, \tag{1.6}$$

- diffeomorphisms infinitesimally generated by the vector field k, if

$$L_k G = 0\,, \qquad L_k F = 0 \Longleftrightarrow \begin{Bmatrix} L_k B = & d\nu \\ L_k A = & dg - \nu \end{Bmatrix} \tag{1.7}$$

  for $g \in C^\infty(M)$ and $\nu \in \Gamma(T^*M)$.

Now we recapitulate the gauged sigma model. The condition on the gauge invariant field strength $F = B + dA$ means that if we write the boundary term in the action as $\int_{\partial\Sigma} X^*A = \int_\Sigma X^*(dA)$ and combine it with the B-field, the action can be gauged via minimal coupling to the abelian gauge field $\mathcal{A}$ with $\delta\mathcal{A} = -d\epsilon$ if $X^a \to X^a + \epsilon k^a$. As was done last time, we assume adapted coordinates $k = \partial/\partial X^0$ and gauge-fix $\mathcal{A}$ to $\mathcal{A} - dX^0$. Then the gauged sigma model is

$$\begin{aligned} S_{\mathrm{g}} = \frac{1}{4\pi\alpha'} \int_\Sigma \Big[& G_{mn}\, dX^m \wedge \star dX^n + F_{mn}\, dX^m \wedge dX^n + 2\, d\lambda \wedge dX^0 \\ & G_{00}\, \mathcal{A} \wedge \star A + 2\, G_{0m}\, \mathcal{A} \wedge \star dX^m + 2\, F_{0m}\, \mathcal{A} \wedge dX^m + -2\, d\lambda \wedge \mathcal{A} \Big] \end{aligned} \tag{1.8}$$

with $m, n \neq 0$ and G as well as F independent of X^0 by (1.7). The dual actions arise as follows.

- As before, we get back to the original model $S(X; G, F)$ by integrating-out λ. This gives $d\mathcal{A} = 0$. The difference to the general discussion of the last talk is that now we already gauge-fixed $\mathcal{A}$ in order to write the gauged sigma model as above. Hence we cannot perform another gauge transformation such that it vanishes, but we solve the EOM by setting

$$\mathcal{A} = dX^0\,. \tag{1.9}$$

  In this way we get back the initial model by identifying the flat gauge field with the initial coordinate along which we dualise. Of course, consistency requires the same treatment of global issues as last time.

- The dual action is obtained by integrating-out $\mathcal{A}$; its EOM is

$$\star A = -\frac{1}{G_{00}} \left(G_{0m} \star dX^m + F_{0m} dX^m + d\lambda \right) . \tag{1.10}$$

Plugging this back into the action and introducing the dual coordinate

$$d\lambda \equiv d\widetilde{X}^0 , \tag{1.11}$$

we obtain the dual sigma model $S(\widetilde{X}; g, f)$ with metric and field strength given by the Buscher rules

$$\begin{aligned} g_{00} = \frac{1}{G_{00}} , \qquad g_{0m} &= -\frac{F_{0m}}{G_{00}} , \qquad g_{mn} = G_{mn} - \frac{G_{m0}G_{0n} + F_{m0}F_{0n}}{G_{00}} \\ f_{0m} &= -\frac{G_{0m}}{G_{00}} , \qquad f_{mn} = F_{mn} - \frac{G_{m0}F_{0n} + F_{m0}G_{0n}}{G_{00}} . \end{aligned} \tag{1.12}$$

Not only the background changes, but also the coordinates X^0 to $\widetilde{X}^0$. Nn-shell the EOM (1.10) give a precise relation: If we recall that $A = dX^0$ solves the λ-EOM and that $\lambda \equiv \widetilde{X}^0$, (1.10) can be rewritten as

$$\boxed{d\widetilde{X}^0 = G_{0a} \star dX^a + F_{0a}\, dX^a} . \tag{1.13}$$

The right-hand side is the conserved current to the isometry $X^0 \to X^0 + \epsilon$, i.e. *T-duality interchanges coordinates with conserved currents.* Let us be more explicit.

1.1 Flat backgrounds

We assume the spacetime to be flat (compact or non-compact) with metric $G = \delta$ and $F = 0$. Then the EOM for the pulled-back coordinates (1.2) become the wave equation $d \star dX^a = 0$. The easiest way to solve it is to introduce light-cone coordinates $\sigma^\pm = \tau \pm \sigma$; then the EOM is $\partial_+ \partial_- X^a = 0$ which is solved by $X^a(\tau, \sigma) = X^a_L(\sigma^-) + X^a_R(\sigma^+)$. In this simple case the coordinate relation (1.13) can be integrated and we see three basic and well-known features of T-duality on these backgrounds:

- **It reflects the right-movers**: The integrated coordinate relation gives

$$\widetilde{X}^0_L = X^0_L \qquad \text{and} \qquad \widetilde{X}^0_R = -X^0_R . \tag{1.14}$$

This shows that for this simple background T-duality reflects the right-moving coordinate; for a general constant background we obtain $\widetilde{X}^0_{L/R} = (F_{0a} \pm G_{0a}) X^a_{L/R}$. The dual background is the same, i.e. $g = \delta$ and $f = 0$.

- **It changes the boundary conditions**: Along X^0 we could either start with Neumann boundar conditions $\partial_\sigma X^0|_{\sigma=0,\pi}$ (since $F=0$) or with Dirichlet boundary conditions $\partial_\tau X^0|_{\sigma=0,\pi}$. Using $\partial_{\tau/\sigma} = \frac{1}{2}(\partial_+ \pm \partial_-)$ they can be translated into conditions on the left- and right-mover with the following effect of T-duality

$$\left.\begin{array}{rl} \partial_- X_L^0 = \partial_+ X_R^0 & \text{(Neumann)} \\ \partial_- X_L^0 = -\partial_+ X_R^0 & \text{(Dirichlet)} \end{array}\right\} \xrightarrow{\text{T-duality}} \left\{\begin{array}{rl} \partial_- \widetilde{X}_L^0 = -\partial_+ \widetilde{X}_R^0 & \text{(Dirichlet)} \\ \partial_- \widetilde{X}_L^0 = \partial_+ \widetilde{X}_R^0 & \text{(Neumann)} \end{array}\right. ,$$

  i.e. T-duality interchanges the boundary conditions in this simple case ($\partial_\tau X^0 \leftrightarrow \partial_\sigma X^0$). This is how D-branes have been discovered: Dirichlet boundary conditions have been neglected because the fixing of string endpoints breaks Lorentz invariance, but with duality they arise inevitably. In general, T-duality along a direction of a Dp-brane gives a D$(p-1)$-brane and a perpendicular T-duality gives a D$(p+1)$-brane.

- **It interchanges momentum and winding**: Since the worldsheet is two-dimensional we can introduce a canonical winding analogous to the canonical momentum. Here this means

$$\begin{aligned} P_0 &= \frac{\partial L}{\partial \partial_\tau X^0} = \frac{-1}{2\pi\alpha'}\,\partial_\tau X^0 \\ W_0 &= \frac{\partial L}{\partial \partial_\sigma X^0} = \frac{1}{2\pi\alpha'}\,\partial_\sigma X^0 \end{aligned} \tag{1.15}$$

  We saw that T-duality exchanges τ- and σ-derivatives. Thus it also interchanges momentum and winding. The contribution of momentum and winding to the zero-modes of the mode expansion of the coordinate fields is via $p_a = \int_0^\pi d\sigma P_a$ and $w_a = \int_0^\pi d\sigma W_a$.

Hence, the coordinate relation (1.13) allows to recover the standard facts about T-duality, but we also see that they are only true in special cases. Generalising the above to arbitrary constant backgrounds is straight-forward. For non-constant backgrounds it is not clear when (1.13) can be integrated and if the result can be interpreted as a coordinate. This is yet another incarnation of the global issues discussed before.

1.2 Example: branes at angles

We follow [1, 2]. Consider the simple background from the last section which is assumed to have dimension 2. Additionally, we place a D1-brane in the (X^0, X^1)-plane of space-time. The easiest way would be to place it along the X^0-direction; this would amount to choosing Dirichlet along X^1 ($\partial_\tau X^1|_{\partial\Sigma} = 0$) and Neumann along X^0 ($\partial_\sigma X^0|_{\partial\Sigma} = 0$). However, the aim is to perform T-duality neither entirely along nor perpendicular to the brane. Therefore we place it at an angle ϕ compared to X^1. This is realised by choosing Neumann boundary conditions along this direction and Dirichlet boundary conditions perpendicular:

$$\begin{aligned}\left(\cos\phi\,\partial_\sigma X^0 + \sin\phi\,\partial_\sigma X^1\right)\big|_{\partial\Sigma} &= 0\\ \left(-\sin\phi\,\partial_\tau X^0 + \cos\phi\,\partial_\tau X^1\right)\big|_{\partial\Sigma} &= 0\,.\end{aligned} \tag{1.16}$$

We perform T-duality along the X^1-direction; this reflects the right-mover, causing the interchange of the τ- and σ-derivative. Then the T-dual boundary conditions are

$$\begin{aligned}\left(\partial_\sigma X^0 + \tan\phi\,\partial_\tau X^1\right)\big|_{\partial\Sigma} &= 0\\ \left(\partial_\sigma X^1 - \tan\phi\,\partial_\tau X^0\right)\big|_{\partial\Sigma} &= 0\,.\end{aligned} \tag{1.17}$$

Comparison with (1.4) shows that this is the boundary conditions in the presence of a field strength

$$F = -\tfrac{1}{2}\,\tan\phi\,dX^0 \wedge dX^1\,; \tag{1.18}$$

thus the D1-brane at angle has become a *magnetized* $D2$-brane by T-duality.

We have just generated a field strength from a pure metric background. As was discussed in the prior section, the Buscher rules do not give a change in background here and in particular no F-field – this seems to contradict the above observation of a field strength appearing in the boundary conditions. However, a constant field strength can (up to global issues) considered as living on the boundary via $F_{ab}dX^a \wedge dX^b = d(F_{ab}X^a dX^b)$.

Symmetric versus asymmetric rotations

There is a nice and handy way of interpreting the change in boundary conditions. Suppose that we start with a D1-brane in X^0-direction, i.e. $\partial_+ X^0_L = \partial_- X^0_R$ and $\partial_+ X^1_L = -\partial_- X^0_R$. Then we obtain (1.16) by a symmetric rotation of left- and right-movers as follows:

$$\begin{aligned}\partial_+\widetilde{X}^0_L &= \partial_-\widetilde{X}^0_R\\ \partial_+\widetilde{X}^1_L &= -\partial_-\widetilde{X}^0_R\end{aligned} \qquad \text{with} \quad \widetilde{\mathbf{X}}_{L/R} = R\,\mathbf{X}_{L/R} \quad \text{and} \quad \mathbf{X}_{L/R} = \begin{pmatrix} X^0_{L/R} \\ X^1_{L/R}\end{pmatrix}\,, \tag{1.19}$$

where the rotation is given by

$$R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\,. \tag{1.20}$$

Then, applying a T-duality gives the magnetised D2-brane as above.

However, we can also perform a T-duality to the D1-brane along X^0 first, to obtain the dual boundary conditions $\partial_+ X^0_L = \partial_- X^0_R$ and $\partial_+ X^1_L = \partial_- X^0_R$ from reflecting X^1_R;

this is a $D2$-brane without flux. We then obtain the same magnetised brane if we now perform an *asymmetric* rotation as follows:

$$\begin{array}{l} \partial_+ \widetilde{X}^0_L = \partial_- \widetilde{X}^0_R \\ \partial_+ \widetilde{X}^1_L = \partial_- \widetilde{X}^0_R \end{array} \qquad \text{with} \quad \widetilde{\mathbf{X}}_L = R\,\mathbf{X}_L \quad \text{and} \quad \widetilde{\mathbf{X}}_R = R^t\,\mathbf{X}_R\,. \tag{1.21}$$

Thus the picture is as follows:

$$\begin{array}{ccc} \begin{array}{c}\text{D1-brane}\\ \text{along } X^0\end{array} & \xrightarrow{\text{LR-symmetric rot.}} & \begin{array}{c}\text{D1-brane}\\ \text{at angle}\end{array} \\ \big\downarrow{\scriptstyle T_{X^1}} & & \big\downarrow{\scriptstyle T_{X^1}} \\ \begin{array}{c}\text{plain}\\ \text{D2-brane}\end{array} & \xrightarrow{\text{LR-asymmetric rot.}} & \begin{array}{c}\text{magnetised}\\ \text{D2-brane}\end{array} \end{array}$$

We conclude that T-duality maps a left-right-symmetric rotation to a left-right-antisymmetric rotation. This is a very convenient way of introducing a field strength into the model.

2 Non-commutative geometry on D-branes

The aim is to study the commutator algebra of the coordinate fields X^0 and X^1 on a magnetised D2-brane via CFT. The first thing we need is the propagator $\langle X^a(z_1), X^b(z_2)\rangle$, but because of the complicated boundary conditions (1.17) it is difficult to derive.[2] However, by the observations made above we can also start from a plain D2-brane, i.e. with simple Neumann boundary conditions for both directions and at both ends of the string and then simply perform an LR-asymmetric rotation (see [1, 2] and the lecture notes [3]).

The propagator is the fundamental solution (or Greens function) to the wave equation respecting the NN boundary conditions $\partial X^a(z)|_{\partial\Sigma} = \bar{\partial} X^a(z)|_{\partial\Sigma}$. The worldsheet Σ is the upper half plane in this picture with the real line its boundary. Thus, skipping details, the method of mirror charges gives the result

$$\begin{aligned} \langle X^a(z_1) X^b(z_2)\rangle &= -\alpha'\,\delta^{ab}\left(\ln|z_1 - z_2| + \ln|z_1 - \bar{z}_2|\right) \\ &= -\tfrac{\alpha'}{2}\,\delta^{ab}\Big[\underbrace{\ln(z_1 - z_2)}_{\langle X^a_L X^a_L\rangle} + \underbrace{\ln(\bar{z}_2 - \bar{z}_2)}_{\langle X^a_R X^a_R\rangle} + \underbrace{\ln(z_1 - \bar{z}_2)}_{\langle X^a_L X^a_R\rangle} + \underbrace{\ln(\bar{z}_1 - z_2)}_{\langle X^a_R X^a_L\rangle}\Big]\,. \end{aligned}$$

The split $X^a(z, \bar{z}) = X^a_L(z) + X^a_R(\bar{z})$ is indicated. Now a constant field strength is introduced easily by the above asymmetric rotation of left- and right-movers via $\widetilde{\mathbf{X}}_L =$

[2]The complex coordinates z arise from the light-cone coordinates by a Wick rotation and exponentiation.

$R\mathbf{X}_L$ and $\widetilde{\mathbf{X}}_R = R^t\mathbf{X}_R$ respectively. With the basic propagators given above we easily obtain

$$\begin{aligned}\langle \widetilde{X}^a(z_1)\widetilde{X}^b(z_2)\rangle =& -\alpha'\,\delta^{ab}\left[\ln|z_1-z_2| - \left(\sin^2\theta - \cos^2\theta\right)\ln|z_1-\bar{z}_2|\right]\\ & -\alpha'\,\epsilon^{ab}\,\sin\theta\,\cos\theta\,\ln\left(\frac{z_1-\bar{z}_2}{\bar{z}_1-z_2}\right)\,.\end{aligned} \tag{2.1}$$

Now the value of the logarithm in the second line depends on the branch cut chosen. Since Σ is the upper half plane, it makes sense to put a branch cut at the negative imaginary axis which makes the propagator single-valued on the upper half plane. Open string vertex operators are inserted at the boundary of the worldsheet; thus we restrict the result to the real line $\partial\Sigma = \mathbb{R}$ via $z_i = \bar{z}_i = t_i$. Also introducing the field strength $\tan\phi = -F_{01} \equiv F$ we obtain

$$\boxed{\langle \widetilde{X}^a(t_1)\widetilde{X}^b(t_2)\rangle = -G^{ab}\,\ln|z_1-\bar{z}_2| + \tfrac{i}{2}\,\theta^{ab}\,\varepsilon(t_1-t_2)}\,, \tag{2.2}$$

where $\varepsilon(t) = \pm 1$ for $t \gtrless 0$ arises from the branch of the logarithm. Moreover, we defined

$$G^{ab} = \frac{2\,\alpha'}{1+F^2}\,\delta^{ab} \qquad \text{and} \qquad \theta^{ab} = 2\pi\alpha'\,\frac{F}{1+F^2}\,\epsilon^{ab}\,. \tag{2.3}$$

The interesting part is the antisymmetric part of the propagator: it distinguishes the order of the inserted operators which causes them to be non-commutative. This can be seen in a sloppy manner by evaluating the commutator at equal positions. Recall that the correlators have to be considered inside the path-integral and radially ordered (this is what time ordering became in the present coordinates). Then we compute

$$\begin{aligned}\langle[\widetilde{X}^0(t),\widetilde{X}^1(t)]\rangle &= \lim_{\delta\to 0}\left[\langle\widetilde{X}^0(t)\widetilde{X}^1(t+\delta) - \widetilde{X}^1(t)\widetilde{X}^2(t+\delta)\rangle\right]\\ &= \tfrac{i}{2}\,\theta^{01} - \tfrac{i}{2}\,\theta^{10} = i\,\theta^{01}\,;\end{aligned} \tag{2.4}$$

thus we have just derived that *the coordinates on equal worldvolume-points of a magnetised D2-brane are non-commutative* with

$$\boxed{[\widetilde{X}^0(t),\widetilde{X}^1(t)] = 2\pi i\alpha'\,\frac{F}{1+F^2}}\,. \tag{2.5}$$

Indeed, it can be shown that the introduction of a constant field strength F on the brane is equivalent to having a pure metric background with the algebra of smooth functions on the brane deformed by the *Moyal-Weyl product*

$$(f\star g)(x) = \exp\left(\tfrac{i}{2}\,\theta^{ab}\,\tfrac{\partial}{\partial x_1^a}\,\tfrac{\partial}{\partial x_2^b}\right) f(x_1)\,g(x_2)\Big|_{x_1=x_2=x}\,, \tag{2.6}$$

provided the *Seiberg-Witten limit* $\alpha' \to 0$ (in a certain manner; see [4]) is taken. Applied to coordinates, this product reproduces the commutator above: $[x^0,x^1] = x^0\star x^1 - x^1\star x^0 = i\theta^{01}$. With this product *the gauge theory on the magnetised brane is equivalent to non-commutative gauge theory on a plane brane.* This was discovered in [4]

In summary we have seen how T-duality applied to branes at angles gives rise to branes with non-commutative worldvolume-coordinates.

References

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