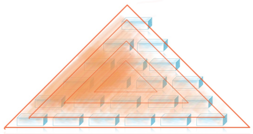
Boson-Lattice Construction for Anyon Models



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This work is an attempt to unveil the skeleton of anyon models. I present a construction to systematically generate anyon models. The construction uses a set of elementary pieces or fundamental anyon models, which constitute the building blocks to construct other, more complex, anyon models. A principle of assembly is established that dictates how to articulate the building blocks, setting out the global blueprint for the whole structure. Remarkably, the construction generates essentially all tabulated anyon models. Moreover, novel anyon models (non-tabulated, to my knowledge) arise. To embody the construction I develop a very physical, visual and intuitive lexicon. An anyon model corresponds to a system of bosons in a lattice. By varying the number of bosons and the number of lattice sites, towers of more and more complex anyon models are built up. It is a Boson-Lattice construction. A self-similar anatomy is revealed: an anyon model is a graph that is filled with bosons to engender a new graph that is again filled with bosons. And further, bosons curve the graph that the anyon model is: I disclose a geography in the space of anyon models, where one is born from another by deforming the geometry of space. I advance an alluring duality between anyon models and gravity.

Anyons are fascinating objects. Their braiding statistics has dramatically shattered our system of beliefs regarding the possible statistics for quantum particles [1–3]. Specially striking is the case of non-Abelian anyons [4–7]: How is it possible that the result of sequentially exchanging pairs in a set of indistinguishable particles might depend on the order in which the exchanges were performed? Anyons emerge as quasiparticles of an exceptional type of organization of matter, which is purely quantum in nature: topological order [8, 9]. In a topologically ordered state a system of individuals constitute a global entity, which acquires a macroscopic self-identity transcending the identities of the microscopic constituents. The set of anyon types emerging from a certain topological order satisfies a collection of *fusion and braiding rules* [10–14], which determine the way in which anyons fuse with each other to give rise to other anyon types, and the form in which they braid around each other. A set of anyon types together with their fusion and braiding rules define an anyon model, which perfectly mirrors its corresponding underlying physical state.

What are the possible anyon models that can exist? From a purely mathematical perspective, it is known that the language underlying anyon models is *modular tensor category* [15–22]. Within this beautiful (and complex) mathematical formalism the answer to this question can be simply phrased: any possible anyon model corresponds to a unitary braided modular tensor category. Wang [13] has tabulated all modular anyon models with up to four anyon types. These correspond, for example, to truncated Lie algebras such as the celebrated Fibonacci model. Bonderson [14] has developed an algorithm to numerically solve the set of algebraic consistency conditions for an anyon model to exist, tabulating a series of very interesting models for up to ten topological charges.

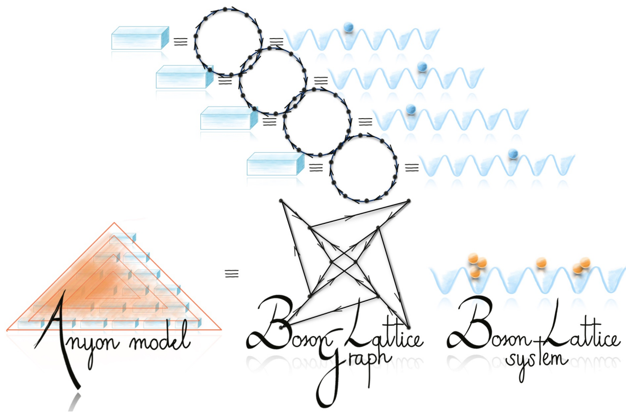
And yet, if we want to delve into our physical knowledge of anyons, we should ask ourselves not only what the possible anyon models are, but, more importantly, what the relational architecture of possible anyon models is. Is there a hidden organization in the set of anyon models? Can we construct complex anyon models from other, simpler ones? Which are the elementary pieces? What is the glue mechanism of these pieces? Moreover, in answering these questions it would be crucial that we use a meaningful physical language. If we are able to capture the essence of anyon mathematical architecture with an intuitive, comprehensive physical vocabulary, we will be closer to understanding the corresponding physical anatomy of topological orders.

In establishing relations between different anyon models, something that we know is how to disintegrate certain complex anyon models into other, simpler ones. This procedure, called *anyon condensation* [23–27], works by making two or more different anyon types become the same. Though no fully general description is known, for the special case in which the condensing anyons have trivial statistics, it is possible to systematically obtain a condensed anyon model from a more complex (uncondensed) one. Little is known, however, about the reverse process, that is, about how to build up more complex anyon models by putting simpler ones together. To go in this "up" direction, we have only straightforward operations at our disposal, such as, for instance, making the product of two or more given anyon models. It is crucial to develop pathways to orderly construct anyon models. This can help us enormously to apprehend the subjacent texture of anyon models and thereby the anatomy of topological orders.

Here, I present a construction to systematically generate anyon models. As in a Lego construction, a set of brick-models are assembled to construct towers of other, more complex, anyon models. The articulation principle follows here a *global blueprint*, the pieces being glued in a correlated manner. To frame the construction I develop a graph language for anyon models. A graph (or a collection of graphs) is used to compactly encode the properties of an anyon model in a visual manner. Topological charges (anyon types) are represented by graph vertices. Fusion rules can be read from the connectivity pattern of the graphs. Braiding rules are obtained through the diagonalization of graphs.

As elementary pieces to begin the construction I choose the Abelian $\mathbb{Z}_n$ anyon models. In the language of graphs these models are represented by periodic one-dimensional oriented lattices, in which each lattice site is connected to its next (to the right) neighbour. Triggered by this graph representation, I make a conceptual leap and identify a $\mathbb{Z}_n$ model with a single particle in a periodic lattice with n sites. This visual image condenses the essence of the building block into *a particle in a lattice*. It inspires the conception of the principle of assembly. This is defined as a *bosonization procedure*, in which particles corresponding to different building blocks are made indistinguishable. The resulting Boson-Lattice system characterizes the constructed anyon model.

I give a prescription to assign a graph to the Boson-Lattice system. The graph is defined as a *connectivity graph* of Fock states. Remarkably, this Boson-Lattice graph always embodies a *modular* anyon model. A dictionary is established between the elements of the Boson-Lattice graph (Fock states, connectivity pattern, eigenvalues and eigenstates), and the properties of the anyon



model (topological charges, fusion rules, braiding rules). By varying the number of building blocks (the number of bosons) and their size (the number of lattice sites), a variety of well known tabulated models arise. Moreover, novel (non-tabulated) anyon models emerge.

The construction reveals a *self-similar* architecture for anyon models. Any anyon model can serve as a building block to engender towers of more complex anyon models. A brick graph is filled with bosons, giving rise to a new graph, which can be again filled with bosons. What is more, the bosons curve the original graph, so that building block graphs and generated graphs are related through a change of metric. This delineates a relational tracery in the space of anyon models, in which anyon models are obtained from each other by deforming the geometry of space.

This article condenses the key essential ideas and results of the Boson-Lattice construction, advancing the pathways it opens up. In [28] I present the full theory, carrying out a complete analysis with thorough proofs, all combined with enlightening illustrations and discussions.

Graph representation for anyon models. An anyon model [10–14] is characterized by a finite set of conserved topological charges or anyon types $\{a, b, \cdots\}$, which obey the fusion rules:

$$a \times b = \sum_c N_{ab}^c \, c, \tag{1}$$

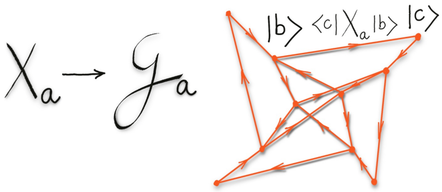
where the multiplicities N_{ab}^c are non-negative integers that indicate the number of ways that charge c can be obtained from fusion of the charges a and b. There exists a unique trivial charge 0 that satisfies $N_{a0}^b = \delta_{ab}$, and each charge a has a conjugate charge $\bar{a}$ such that $N_{ab}^0 = \delta_{b\bar{a}}$. Meanwhile, the charges obey a set of braiding rules, encoded in the so called topological S and $\mathcal{T}$

matrices, that determine the way in which anyons braid around each other and around themselves. For an anyon model to exist, fusion and braiding rules have to fulfill a set of of consistency conditions, known as the Pentagon and Hexagon equations.

Let me consider a Hilbert space of dimension equal to the number of topological charges in the anyon model. I denote the canonical basis in this Hilbert space by $\{|a\rangle, |b\rangle, \ldots\}$, where each state is associated with a charge in the anyon model. I define an algebra of operators $\{X_a, X_b, \cdots\}$ with matrix elements:

$$\langle c|X_a|b\rangle = N_{ab}^c. \quad (2)$$

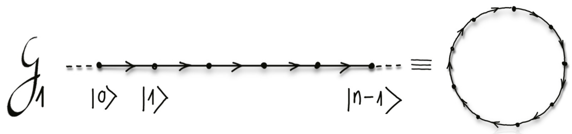
This *topological algebra* defines a set of *mutually commuting and normal* operators. The operator corresponding to the trivial charge is the identity operator, and conjugate charges are associated with adjoint operators, $X_{\bar{a}} = X_a^\dagger$.



I represent each operator in the algebra by a weighted directed graph $\mathcal{G}_a$, which I call *topological graph*. The vertices of the graph are in one to one correspondence to the states of the canonical basis. Two vertices $|b\rangle$ and $|c\rangle$ are connected if the matrix element $\langle c|X_a|b\rangle$ is different from zero. The link is assigned a weight N_{ab}^c, and represented by a N_{ab}^c-multiple line oriented from $|b\rangle$ to $|c\rangle$.

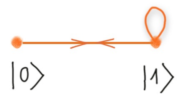
Within the graph language, anyon models can be represented in a visual and illuminating way. Fusion rules are encoded in the connectivity pattern of the graphs. Moreover, braiding rules can be obtained through their simultaneous diagonalization. The topological S-matrix, encoding the anyon mutual statistics, is formed by a set of common eigenvectors of the topological graphs. Furthermore, for elementary anyon models (those that are not the product of two or more anyon models), a single graph in the algebra is able to encode the complete set of graphs [1]. This *generating graph* is a *connected* graph: there is a path of links connecting any pair of vertices.

For instance, let me consider the Abelian anyon model $\mathbb{Z}_n$, which is characterized by a set of n charges: $\{0, 1, \cdots, n-1\}$, with fusion rules $a \times b = a + b \,(\bmod\ n)$. The fusion and braiding rules of this model can be compactly encoded in the graph corresponding to the charge 1: an *oriented one-dimensional lattice with periodic boundary conditions.*



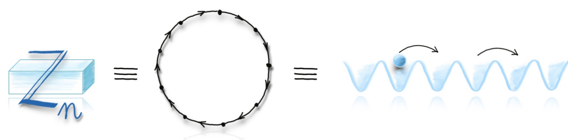
The topological algebra can be directly read from the connectivity of the graph as $\{\mathbb{1}, X, \cdots, X^{n-1}\}$, where X is the chiral translation operator in the ring, satisfying $\langle b|X|a\rangle = \delta_{b,a+1}$, and $X^n = \mathbb{1}$. Moreover, by trivially diagonalizing this graph, and realizing that the eigenvectors are plane waves of the form $\langle a|\psi_b\rangle = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n}a \cdot b}$, we directly obtain the S-matrix of the $\mathbb{Z}_n$ model.

Another seminal model, the Fibonacci model can be embodied in the graph below.



From the graph we read that the model is characterized by two charges $\{0, 1\}$. The topological algebra is $\{\mathbb{1}, X\}$, with $X^2 = \mathbb{1} + X$. Diagonalization of the graph directly gives us the quantum dimension $d = \varphi$ and the topological spin $\theta_1 = e^{i4\pi/5}$ of the non-trivial charge 1, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

Building blocks. Any elementary anyon model can serve as a building block for the construction. The first key idea is to identify the building block with its generating graph. What is more, I make an abstraction and condense the anyon model into a physical image. The building block is identified with a particle "moving" in a lattice as dictated by the connectivity pattern of the generating graph. For example, a $\mathbb{Z}_n$ anyon model corresponds to a particle in an oriented ring, hopping from one site to the next to the right. This image captures the essence of the elementary pieces of the construction. The building blocks are particles in lattices. The construction will assemble particles in lattices.



[1] An anyon model that is the product of m elementary models will be encoded in m generating graphs, one per model.

Boson-Lattice construction. I give now a prescription to assemble the building blocks in order to sequentially generate new anyon models. I present the formalism for $\mathbb{Z}_n$ building blocks, and then generalize it to any brick model. The extremely simple $\mathbb{Z}_n$ Abelian models are able to generate essentially (up to trivial operations such as products or embeddings) all tabulated anyon models.

To construct a new anyon model I consider k identical building blocks of length n. I define the Hilbert space $\mathcal{H}(k,n)$ associated with the new anyon model as the one resulting from bosonization (symmetrization) of the tensor product of the k identical Hilbert spaces of the building blocks:

$$\mathcal{H}(k,n) = \mathcal{S}\ \underbrace{\mathcal{H}(1,n) \otimes \cdots \otimes \mathcal{H}(1,n)}_{k \text{ copies}}. \tag{3}$$

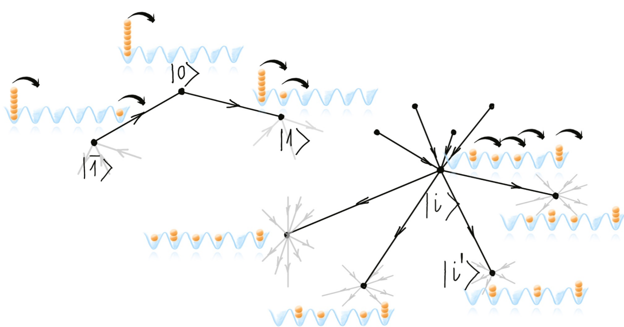
The Hilbert space $\mathcal{H}(k,n)$ is that of k bosons in a ring of n lattice sites. This defines the Boson-Lattice system. I consider the basis of Fock states. Each Fock state is characterized by the corresponding occupation numbers of the lattice sites:

$$|i\rangle \equiv |n_0^{(i)}, n_1^{(i)}, \cdots, n_{n-1}^{(i)}\rangle\,, \tag{4}$$

with $n_\ell^{(i)}$ being the occupation number of site ℓ, $\ell = 0, \cdots, n-1$, and $\sum_\ell n_\ell^{(i)} = k$.

Boson-Lattice Graph. Associated with the Hilbert space $\mathcal{H}(k,n)$ I define a graph $\mathcal{G}(k,n)$ which will be the generating graph of the constructed anyon model. The graph is defined as follows:

- The vertices of the graph are in one to one correspondence with the Fock states in the Boson-Lattice system.
- Two vertices are connected if the corresponding Fock states are connected by tunneling of one boson to the next (to the right) lattice site.
- The link is given a weight 1.



The central result of this work states that *the Boson-Lattice graph defined above is the generating graph of a modular anyon model*. Important to emphasize is, that the constellation of conditions that a graph has to fulfill in order to be the generating graph of a modular anyon model is so restrictive that a randomly chosen graph would have low chances to succeed. Startlingly, the special connectivity properties of the graph I have defined guarantee the existence of a modular anyon model, for any number of bosons, for any number of lattice sites. This anyon model has the following properties.

Topological charges. The *topological charges* of the Boson-Lattice anyon model are in one to one correspondence with the *Fock states* of the Boson-Lattice system. An anyon type is thereby represented by a Fock state. The trivial charge 0 corresponds to the Fock state $|0\rangle \equiv |k, 0, \cdots, 0\rangle$, with all bosons occupying the same lattice site $\ell = 0$. The generating charge of the model will be denoted by 1 and is represented by the Fock state $|1\rangle \equiv |k-1, 1, \cdots, 0\rangle$, obtained from the state $|0\rangle$ by transferring one boson to site 1. Given a charge a, the conjugate charge $\bar{a}$ is represented by the conjugate Fock state, $|\bar{a}\rangle$, whose occupation numbers are obtained by reflection with respect to the site 0, $n_\ell^{(\bar{a})} = n_{n-\ell}^{(a)}$.

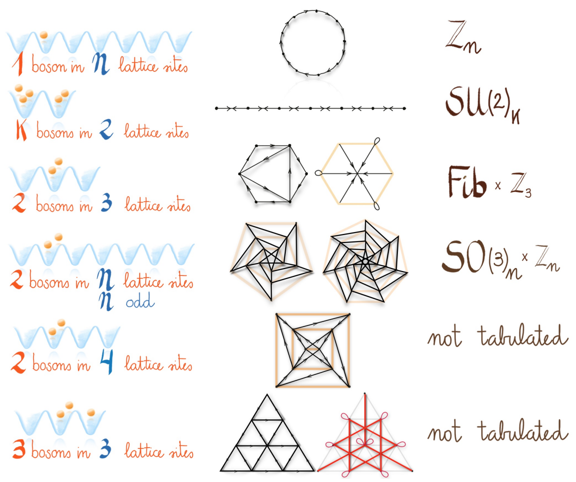
Topological algebra and fusion rules. The fusion rules of charge 1 are directly extracted from the connectivity pattern of the Boson-Lattice graph. Denoting by X the many-body operator corresponding to the graph, we have

$$1 \times a = \sum_b \langle b|X|a\rangle\, b. \tag{5}$$

The special features of the Boson-Lattice graph assure that the operator X can be always completed to a topological algebra. First, the graph is *connected*, since there is always a sequence of consecutive tunneling moves of one particle to the next (to the right) lattice site that connects two arbitrary Fock states. Also, the graph is *normal*: it commutes with its adjoint, $X^\dagger$, a graph with the same links, but arrows reversed. Moreover, the graph is invariant under the chiral global translation T, which translates all particles by one site. These properties guarantee that for each charge a there exists a unique operator X_a of the form:

$$X_a = \mathsf{p}_a(X, X^\dagger, T), \tag{6}$$

where p_a is a polynomial of integer coefficients of the operators $X, X^\dagger$ and T satisfying: $X_a\,|0\rangle = |a\rangle$ and $\langle c|X_a|b\rangle = 0, 1, 2, \cdots$. This set of polynomials defines the topological algebra of the anyon model, whose fusion rules are given by $N_{ab}^c = \langle c|X_a|b\rangle$.

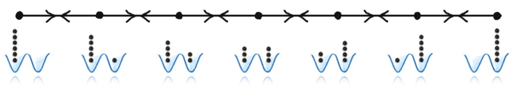


Examples of anyon models generated by the construction, using the Abelian anyon models $\mathbb{Z}_n$ as building blocks.

Quantum dimensions and modular matrices. Since the Boson-Lattice graph is connected, the Perron-Frobenius theorem guarantees the existence of an eigenvector with all positive components. These components define the quantum dimensions d_a of the anyon model. The characteristic features of the graph assure that its eigenvectors $\{|\psi_b\rangle\}$ can be chosen to compose a unitary and symmetric matrix $S_{ab} = \langle a|\psi_b\rangle$, which defines the topological S-matrix of the anyon model. Furthermore, the graph is such that this symmetric matrix fulfills the modular relation $(S\mathcal{T})^3 = \Theta S^2$, where $\mathcal{T}_{ab} = \theta_a \delta_{ab}$ and $\Theta = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a = e^{i2\pi c/8}$. The diagonal matrix $\mathcal{T}$ defines the $\mathcal{T}$-matrix of the Boson-Lattice model, θ_a are the topological spins, and c is the central charge.

By assembling different numbers of identical building blocks (by varying the number of bosons k) and by changing their size (varying the number of lattice sites n) the construction succeeds in generating towers of well known tabulated anyon models. To see the construction at work, I have selected three illustrative examples.

***k* bosons in 2 lattice sites**. By assembling k copies of the Abelian building block $\mathbb{Z}_2$, the construction generates the non-Abelian model $\mathbf{SU}(2)_k$. The Boson-Lattice system is that of k bosons in a lattice of 2 sites, and has dimension $k+1$. Following the prescription of the construction, the Boson-Lattice graph corresponds to a one-dimensional lattice of $k+1$ sites in which each vertex is connected to its two next neighbors. The two ending vertices are not connected to each other.

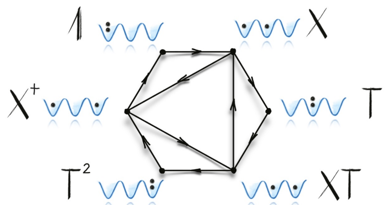


The corresponding operator is

$$X = \sum_{x=0}^{k-1} |x+1\rangle \langle x| + \text{h.c.}, \tag{7}$$

which corresponds to the (real) tunneling operator of one particle in a one-dimensional lattice of $k+1$ sites with open boundary conditions. Diagonalization of the operator X is trivial. The eigenmodes are sine functions that vanish at the boundaries of the one-dimensional lattice. They have the form: $|\psi_q\rangle = \sum_x S_{qx} |x\rangle$, where $S_{qx} = \frac{\sqrt{2}}{\sqrt{k+2}} \sin \frac{\pi}{k+2}(q+1)(x+1)$. Pleasingly, they define a unitary symmetric matrix that exactly corresponds to the S-matrix of the $\mathbf{SU}(2)_k$ anyon model.

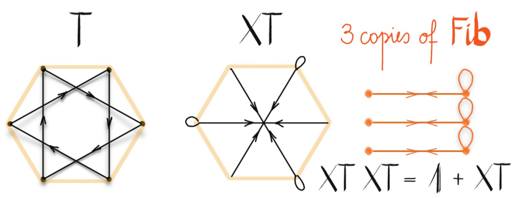
2 bosons in 3 lattice sites. By assembling 2 copies of the Abelian building block $\mathbb{Z}_3$, the construction generates the non-Abelian model **Fib** $\times\, \mathbb{Z}_3$. This anyon model is connected to the non-Abelian Read-Rezayi state for the $\nu = 12/5$ quantum Hall plateaux [14]. The Boson-Lattice system is that of 2 bosons in a lattice of 3 sites, which has dimension 6. The corresponding anyon model has thus 6 topological charges. The Boson-Lattice generating graph is the one depicted below.



By inspection of the graph, the set of polynomials forming the topological algebra are obtained as:

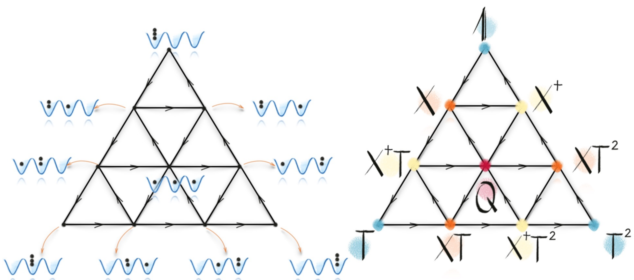
$$\mathcal{A} = \{\mathbb{1}, X, T, XT, T^2, XT^2\}. \tag{8}$$

It is illuminating to draw the graph corresponding to the operator XT, which decomposes into three identical copies of the generating graph of the Fibonacci model.



Diagonalization of the generating graph X yields a unique (up to conjugation) unitary symmetric matrix of eigenvectors. This matrix is the tensor product: $S = S_{\mathsf{Fib}} \otimes S_{\mathbb{Z}_3}$, where S_{Fib} and $S_{\mathbb{Z}_3}$ are, respectively, the S-matrix of the Fibonnaci and the $\mathbb{Z}_3$ anyon models. The corresponding $\mathcal{T}$-matrix is $\mathcal{T} = \mathcal{T}_{\mathsf{Fib}} \otimes \mathcal{T}_{\mathbb{Z}_3}$.

3 bosons in 3 lattice sites. By assembling 3 copies of the Abelian building block $\mathbb{Z}_3$, the construction generates a well defined non-Abelian anyon model with 10 anyon types. This model is not tabulated. The Boson-Lattice system is that of 3 bosons in a lattice of 3 sites, which has dimension 10. The Boson-Lattice graph is the one depicted below.



By inspection of the graph the set of polynomials composing the topological algebra is derived as:

$$\mathcal{A} = \{\mathbb{1}, T, T^2, X, XT, XT^2, X^\dagger, X^\dagger T, X^\dagger T^2, Q\},$$

where the operator Q fulfills: $Q = XX^\dagger - \mathbb{1}$ and $QT = Q$. Interestingly, this model has multiplicities larger than 1. We have:

$$Q \times Q = 1 + T + T^2 + 2Q. \tag{9}$$

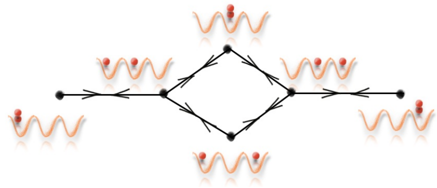
This model is not tabulated in the tables by Bonderson [14], which are restricted to multiplicity-free anyon models. Diagonalization of the graphs yields a unique (up to conjugation) symmetric and unitary matrix, which defines the S-matrix of the anyon model. The topological $\mathcal{T}$-matrix is uniquely determined by the modular relation above, relating the S and $\mathcal{T}$ matrices.

Self-similarity of the construction. Anyon models generated by assembling the building blocks $\mathbb{Z}_n$ can be used themselves as building blocks to generate new models at higher levels of the construction. In fact, any anyon model can serve as a building block to start the construction. The principle of assembly can be generally formulated like this.

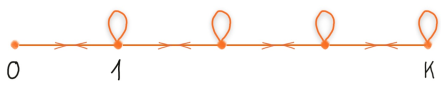
Given k copies of a building block with generating graph $\mathcal{G}$, the Boson-Lattice system is the one of k bosons in that graph.

The Boson-Lattice graph $\mathcal{G}_{\mathcal{B}}$ is defined as the connectivity graph of Fock states of the Boson-Lattice system, where two Fock states are linked if they are connected by a one-particle move allowed by the building block graph $\mathcal{G}$.

This defines the self-similar blueprint to generate a graph from a graph, an anyon model from an anyon model. For example, assembling two copies of the model $\mathbf{SU}(2)_2$, which was generated itself by assembling two copies of a $\mathbb{Z}_2$ model, we obtain the Boson-Lattice graph,



which corresponds to the anyon model $\mathbf{SO}(5)_2$. As another interesting example, assembling k copies of the Fibonacci model, we obtain the graph



which corresponds to the anyon model $\mathbf{SO}(3)_{2k+1}$.

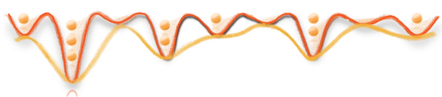
Curved geography of anyon models. The Boson-Lattice graph is constructed by filling with bosons the graph of the building block. Whereas the building block graph can be represented by a one-particle operator

$$X_{\mathcal{G}} = \sum_{x \rightsquigarrow x' \in \mathcal{G}} a^\dagger_{x'} a_x$$

(with the operator $a^\dagger_x (a_x)$ creating (annihilating) a particle at site x, and $x \rightsquigarrow x'$ indicating that there is an oriented link from x to x' in the graph $\mathcal{G}$), the Boson-Lattice graph corresponds to a many-particle operator, $X_{\mathcal{G}_{\mathcal{B}}}$. There is an illuminating form of writing it. It is a *correlated tunneling* operator in which a particle tunnels according to the connectivity of the building block graph $\mathcal{G}$, but, crucially, with an amplitude that depends on the local density of bosons:

$$X_{\mathcal{G}_{\mathcal{B}}} = \sum_{x \rightsquigarrow x' \in \mathcal{G}} a^\dagger_{x'}\, g(n_x, n_{x'})\, a_x .$$

Effectively, the bosons curve the metric of the original graph. I can say that the construction builds the new graph by curving the metric of the building block graph.



This discloses a fascinating structure: an anyon model is engendered from another by deforming the geometry of its corresponding space. Moreover, this points to a hidden symmetry of the Pentagon and Hexagon equations: they seem to be invariant under such metric deformations. And therein lies the success of the construction: starting with a well defined anyon model, the *curvature* induced by the construction, changes the properties of the model while nicely preserving the intricate set of consistency conditions, so that a more complex, but still well defined anyon model emerges.

Closures and openings. The Boson-Lattice construction provides an orderly systematic way to construct anyon models. By assembling different numbers of identical building blocks (by varying the number of bosons k) and by changing their size (varying the number of lattice sites n) the construction succeeds in generating towers of well known tabulated anyon models. The construction reveals a self-similar skeleton for the space of anyon models: anyon models are grouped in levels of hierarchy, with models at a certain level being the building blocks of those at the next, the articulation principle repeating itself as a bosonization procedure that makes the elementary pieces indistinguishable.

The Boson-Lattice construction represents the collapse of two languages into one. It reveals that the mathematical language to describe anyon models can be the same as the one describing boson-lattice systems. It provides us with a physically meaningful language (the one of bosons, Fock states and tunneling Hamiltonians) to describe the mathematical properties (fusion rules and braiding rules) of anyon models. I believe this physical language can help us in our way to fill the explanatory gap between the mathematical and the physical sides of topological orders.

I have focused in this work on the construction of *modular* anyon models, for which corresponding conformal field theories and topological quantum field theories exist. Can we delineate a graph of correspondences between the Boson-Lattice construction and conformal and topological quantum field theories? Might the Boson-Lattice construction bring light into our understanding of the anatomy of conformal field theories and topological quantum field theories?

The Boson-Lattice construction can be enriched by adding internal degrees of freedom to the bosons participating. For example, we can consider bosons with color or spin, or also change the dimensionality of the lattice. In this way, anyon models corresponding to non-chiral topological orders, such as *quantum doubles*, can be generated. Moreover, the construction can be extended to fermions. By considering Fermion-Lattice graphs other towers of anyon models are built up.

In light of the Boson-Lattice approach, the reverse process of disintegrating anyon models into simpler pieces acquires an enlightened perspective. Within the Boson-Lattice picture, anyon condensation is a condensation of actual bosons. Moreover, the concept of topological symmetry breaking corresponds to actual symmetry breaking in the Boson-lattice system. I find extremely interesting to investigate how anyon condensation is precisely described in the Boson-Lattice language, and, moreover, whether the Boson-Lattice construction can guide us to develop a systematic framework to describe anyon condensation, for which no fully general description is known.

Especially interesting is the prospect of developing a dual construction at the microscopic physical level. I believe the Boson-Lattice construction for anyon models can inspire the blueprint of a construction of many-body wave functions and Hamiltonians for the corresponding topological orders. A dual bosonization principle of assembly could be used to orderly build up complex topologically ordered many-body wave functions from elementary ones. Such systematic framework would reveal a dual anatomy in the phase space of topologically ordered systems.

I have drawn a correspondence between anyon models and curved-space geometries. What type of space-geometries do anyon models correspond to? How can time be included into the theory? I believe that this connection anticipates a beautiful duality between anyon models and gravity, which I feel compelled to unveil.

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